Recent Work in Shape-Based Methods for Diffusive Inverse Problems

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(Dated: August 14, 2002)

Abstract

We consider geometric inversion methods designed to directly determine information concerning the size, shape, location, and perhaps number of anomalies in a region of interest [1, 2] for diffusive inverse problems arising in medical imaging and environmental remediation. First, a parametric approach to the problem is derived and validated using a diffuse optical tomography (DOT) sensing example. A second technique to identify boundaries of an unknown number of objects is based on the idea of curve evolution [2]. This approach mathematically "shrink-wraps" a deformable surface in 3D or curve in 2D to the boundary of one or more objects. We demonstrate the utility of this method using an electrical resistance tomography (ERT) sensing example in three dimensions.

INTRODUCTION AND BASIC METHODOLOGY

For the sensing problems of interest here, a collection of transmitters emit a form of time-harmonic energy into a medium. The medium is composed of a nominal background and a collection of localized, anomalous structures. After interacting with the medium, the diffused energy is observed by a collection of detectors also arrayed on the boundary of the medium. For DOT [3], laser light modulated up to a couple of hundred MHz is transmitted into tissue using optical fibers. Fibers are also used to measure the scattered light. Because tissue is turbid, the photons obey a transport equation which, in the limiting case valid for most all DOT applications, is approximated by a diffusion equation. Hence for DOT the data are in the form of diffuse photon density waves. For ERT [4], electrical current at zero frequency is injected into the medium and the receivers measure voltages along a portion of the boundary.

For the class of problems of interest here, the placement of the sources and detectors is limited. In many cases, they are restricted to lie on one side of the medium (a *reflection* geometry). In other cases, the sources are on one side and the receivers on another (a *transmission* geometry). Sometimes it is possible to position the sources on one side and the receivers on the two adjacent sides, similar to vertical seismic profiling in geophysical exploration. In any event for DOT, ERT and many other applications, it is not possible to fully encircle the unknown medium, leaving us with a limited view tomographic inverse problem. Such problems are known to be highly ill-posed making the recovery of a dense set of pixel or voxel values a very delicate procedure. Hence we are motivated to explore methods designed to recover reduced complexity, geometric models of the unknown the structure of which are more closely tied to the underlying anomaly characterization goal of the sensing problem

The physical model for diffuse wave imaging is a frequency domain diffusion equation:

$$\nabla \cdot \sigma(r)\nabla\phi(r) + k(r,\omega)\phi(r) = \text{sources} \tag{1}$$

with ω the source modulation frequency. The relationship of σ and k to the physical properties of the medium, the physical significance of ϕ, and the boundary conditions are all application dependent. For DOT, $\sigma(r) = 1/(3\mu_{s'}(r))$, $k(r) = \mu_a(r) + j\omega/c$ [3] and ϕ is the diffuse photon density wave. Here, μ_a, the optical absorption coefficient and $\mu_{s'}$, the reduced scattering coefficient, are the physical parameters of interest. Finally, Robin boundary conditions are employed. For ERT, $k = 0$ since we are probing at DC and σ, the electrical conductivity, is the desired quantity. In this case, ϕ is the electrical potential and Neumann boundary conditions are used at the air-medium interface[4].

Given the physical model embodied in (1), the processing goal of specific interest in this paper is to identify the geometric structure of anomalous regions in the space varying structure of the physical parameters of interest for the intended application. Specifically, for ERT, we concentrate on identifying σ while for the DOT we consider the case where only perturbations in the absorption are to be characterized. In each case we seek estimates of parameter which minimize a least squares error functional of the form:

$$\mathcal{E} = \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{N}\left|w_{ij}(\phi^i(\mathbf{r}_j) - \phi^i_{\text{obs}}(\mathbf{r}_j))\right|^2. \tag{2}$$

Here, $\phi^i(\mathbf{r}_j)$ is the hypothesized sensor response computed via a computational forward model due to source i and detector j, and $\phi^i_{\text{obs}}(\mathbf{r}_j)$ is the actual, observed response. The weight w_{ij}, if strictly real and usually related to the inverse of the variance of the noise for that sensor.

A key element of any gradient based algorithm (e.g. steepest decent or non-linear conjugate gradient) for minimizing $\mathcal{E}$ is the efficient computation of the functional gradient, or Fréchet derivative, of $\mathcal{E}$ with respect to a change in either σ or k. In the case of k, as shown in [5] this can be accomplished as $\nabla_{\mu_a(\mathbf{r})}\mathcal{E}(\mathbf{r}) = \sum_{i=1}^{M} \text{Re}\left[\tilde{\phi}^i(\mathbf{r})\phi^i(\mathbf{r})\right]$ where $\tilde{\phi}^i(\mathbf{r})$, the adjoint field due to source i, satisfies the adjoint PDE:

$$-\nabla\cdot\nabla\sigma(\mathbf{r})\tilde{\phi}^i(\mathbf{r}) + (\mu_a(\mathbf{r}) + \frac{j\omega}{c})\tilde{\phi}^i(\mathbf{r}) = \tilde{s}(\mathbf{r}) \tag{3}$$

where the adjoint source is defined by

$$\tilde{s}(\mathbf{r}) = \sum_{j=1}^{N}\left[w_{ij}(\phi^i(\mathbf{r}_j) - \phi^i_{\text{obs}}(\mathbf{r}_j))\right]^*\delta(\mathbf{r}-\mathbf{r}_j) \tag{4}$$

with v the speed of light in the medium.

For problems in which we seek to determine σ, the sensitivity matrix associated with the ith source is $A_i = \left[\frac{\partial\phi^i}{\partial\sigma(\mathbf{r}_j)}\right]$; i.e. the Jacobian matrix obtain by taking the derivatiove of each element of the ith observation vector with respect to the values of the conductivity in every voxel under consideration. As we discuss below, the quantity required in our calculation is $A_i(\sigma)^T(\overline{\phi}^i_{obs} - \overline{\phi}^i)$. To efficiently carry out this computation, we consider the discrete representation of the Poisson equation, $\mathbf{G}\overline{\phi}^i = \overline{s}^i$ and following the work in [4] use the representation

$$A_i(\sigma)^T(\overline{\phi}^i_{obs} - \overline{\phi}^i) = -\begin{pmatrix} (\overline{\phi}^i)^T \frac{\partial \mathbf{G}}{\partial \sigma_1} \\ \vdots \\ (\overline{\phi}^i)^T \frac{\partial \mathbf{G}}{\partial \sigma_L} \end{pmatrix} \mathbf{G}^{-1}(\overline{\phi}^i_{obs} - \overline{\phi}_i). \tag{5}$$

Since $\mathbf{G}$ is a discretized version of a PDE, it is a sparse matrix as are the derivatives required in (5). Finally, the adjoint field for the ith source is given the term $\tilde{\phi}^i \equiv \mathbf{G}^{-1}(\overline{\phi}^i_{obs} - \overline{\phi}_i)$.

DIFFUSE OPTICAL TOMOGRAPHY

Here we consider a parametric estimation algorithm, in which the absorption distribution is assumed to take the form of a sphere with center (x_o, y_o, z_o), radius r and absorption μ_a^o in a known background μ_a^b:

$$\begin{aligned} \mu_a(\mathbf{r}) &= \mu_a^b(\mathbf{r}) + (\mu_a^o(\mathbf{r}) - \mu_a^b(\mathbf{r}))H(O(\mathbf{r})) \\ O(\mathbf{r}) &= r^2 - (x - x_o)^2 - (y - y_o)^2 - (z - z_o)^2 \end{aligned} \tag{6}$$

where $H(\mathbf{r})$ is the 3-D Heaviside step function, and $O(\mathbf{r})$ is the object shape function. We use a typical steepest decent approach to solve for the parameters where the gradient of the error with respect to e.g. x_o is:

$$\frac{\partial \mathcal{E}}{\partial x_o} = \int\int\int 2(x - x_o)\delta(O(\mathbf{r}))\nabla_{\mu_a(\mathbf{r})}\mathcal{E}\, d^3\mathbf{r} \tag{7}$$

with $\nabla_{\mu_a(\mathbf{r})}\mathcal{E}\, d^3\mathbf{r}$ obtained from (3).

As an example, we consider the case of determining the location, radius, and contrast of a spherical absorbing inhomogeneity within a cube with dimensions $6 \times 6 \times 6$cm. A reflection geometry is used, with sources and detectors interlaced on the $z = 0$ plane. The background absorption is 0.05cm^{-1}, while that of the absorbing sphere is 0.3cm^{-1}, and a value of 15cm^{-1} for μ_s was used. In Fig. 1 we show the progression of the estimates for the object center as function of algorithm iteration. We start from an initial point in the parameter space and, at each iteration, conduct a line search in the direction of the gradient. Iteration 10, the radius had converged to slightly under 1 cm and the contrast to about 0.29 cm^{-1}.

RESISTANCE TOMOGRAPHY

For the ERT problem we assume that the medium consists of two distinct regions, the background and an inclusion, and use surface-evolution methods to identify the boundary of the inclusion. Denote by, D the boundary of the inclusion with conductivity σ_{int}. Similarly, σ_{ext} is the conductivity of the background. Let $\hat{\partial \mathrm{D}}$ be an estimate of $\partial\mathrm{D}$, the boundary of D. Let $\psi(\mathbf{r})$ be a level set function such that $\hat{\partial \mathrm{D}} := \{\mathbf{r}; \psi(\mathbf{r}) = 0\}$. The ERT inverse problem can be formulated as determining the minimum of $\mathcal{E}$ over all ψ where $\sigma(\mathbf{r}) = \sigma_{int}$ for $\psi(\mathbf{r}) < 0$ and $\sigma(\mathbf{r}) = \sigma_{ext}$ for $\psi(\mathbf{r}) \geq 0$. The evolution of the level-set function ψ, in artificial time, is defined by the Hamilton-Jacobi equation of the form :

$$\partial_t \psi + \alpha(t)\mathbf{n} \cdot |\nabla\psi| = 0. \tag{8}$$

where $\alpha(t)$ is the velocity of the evolving contours in their outward normal direction $\mathbf{n}$ [6].

In most all previous work in this area, the velocity function is $\alpha(t) = \mp \sum_{i=1}^{N} A(\sigma)^T(\phi^i_{obs} - \phi^i)$, with A the Jacobian matrix and the sum is over the observation vectors generated by N sources. Unfortunately the physics of the problem is such that this choice of velocity causes the level-set surface to move quickly in regions of space close to the sources and receivers and far slower elsewhere. In other words, the sensitivity of $\mathcal{E}$ to changes in ψ is artificially low especially in those areas of space where we desire information.

To overcome this difficulty, we propose multiplying the previously defined velocity function by a positive diagonal matrix, $\mathbf{P}(t)$. For pixels with significant domain derivative, the corresponding diagonal entry of $\mathbf{P}(t)$ is defined as follows:

$$[\mathbf{P}(t)]_{l,l} = |\alpha(l)|^{-1}\left(\mu.\alpha_{max} + (1-\mu)|\alpha(l)|\right), \tag{9}$$

where α_{max} is the maximum, over all l, of $|\alpha(l)|$ and μ is a parameter smaller but close to 1. As we discuss in the talk, this choice maintains the direction of the velocity at each point while "equalizing" the speed dynamic range to lie between $\mu\alpha_{max}$ and α_{max}.

We simulate a 3-D binary medium with two disjoint cylindrical inhomogeneities, Fig. 2(a). We initiate our algorithm with a single sphere occupying basically the entire region. The final result, shown in Fig. 2(b), demonstrates the capability of the algorithm to reconstruct the unknown shape even for a poor initial guess.

CONCLUSIONS

In this paper, we have presented the basic formations and initial examples of geometric methods for the solution of inverse problems using diffuse wave data. Methods employing both parametric and non-parametric parameterizations of the unknown have been considered. At the heart of both approaches is the use of adjoint field techniques for sensitivity calculation. Our future work in this area is focused on the validation of these methods using real sensor data as well as the extension of the approaches to lift some of the simplifying assumptions.

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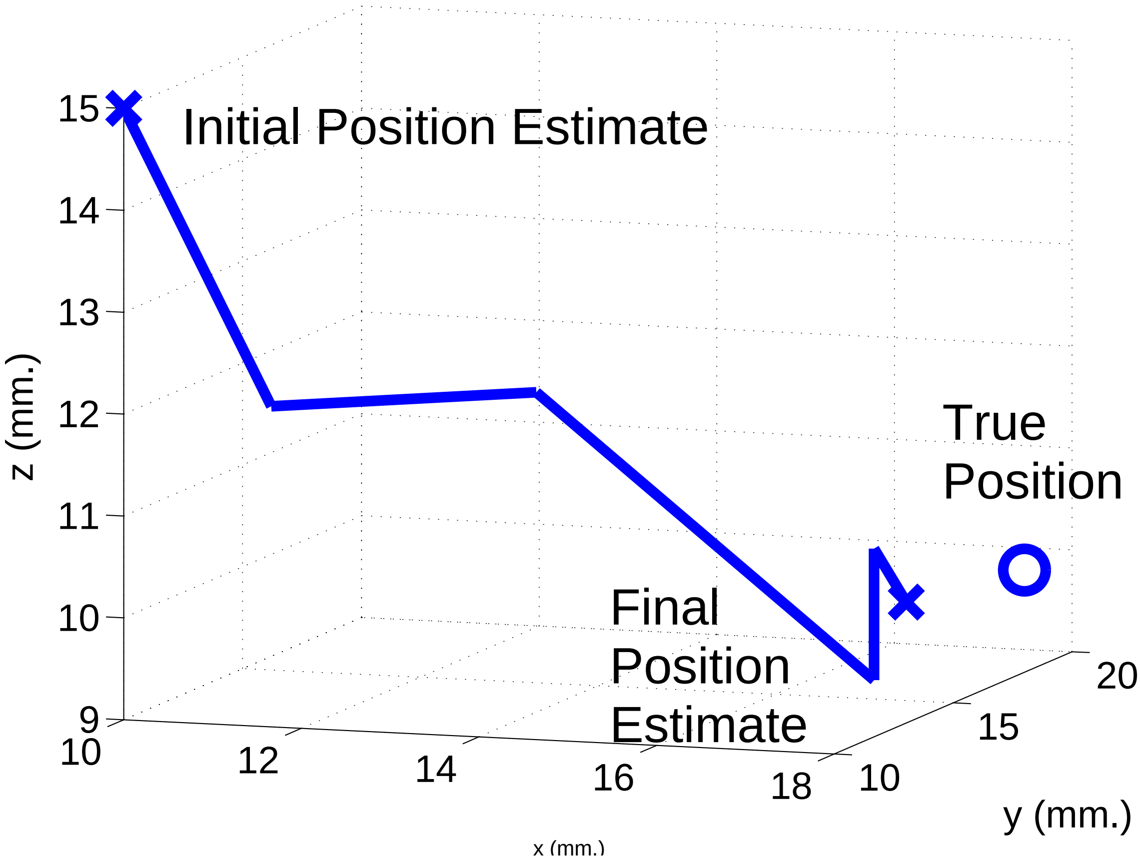
FIGURE CAPTIONS

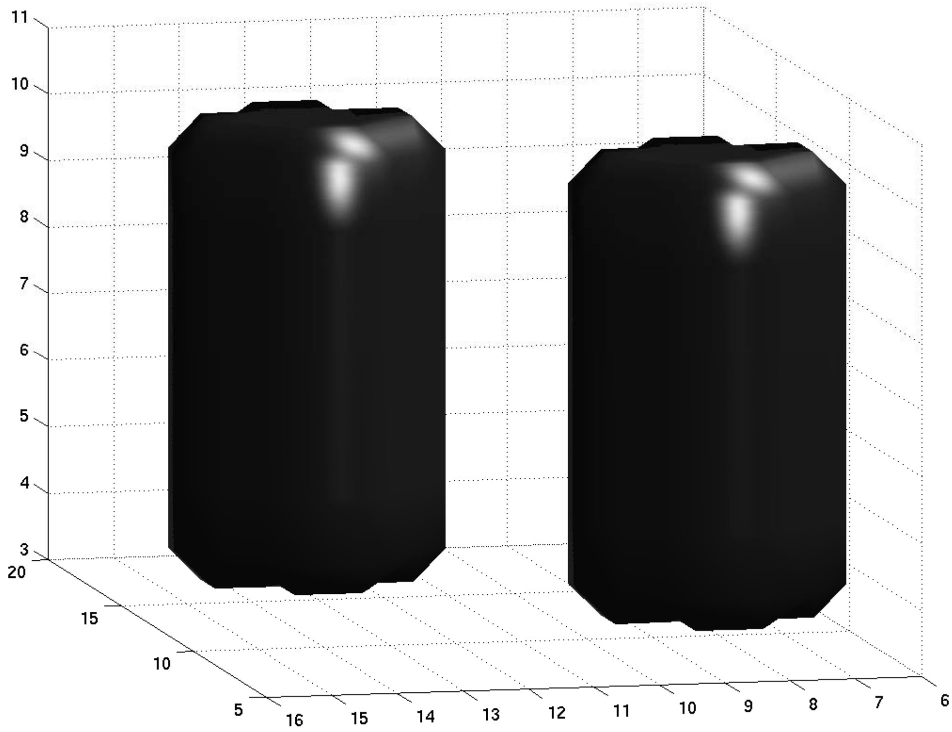
1. **Figure 1** Shape-based inversion results for DOT problem. Center of hypothesized absorber at each iteration

2. **Figure 2** Shape-based inversion results for ERT problem. 1 voxel=1 m^3

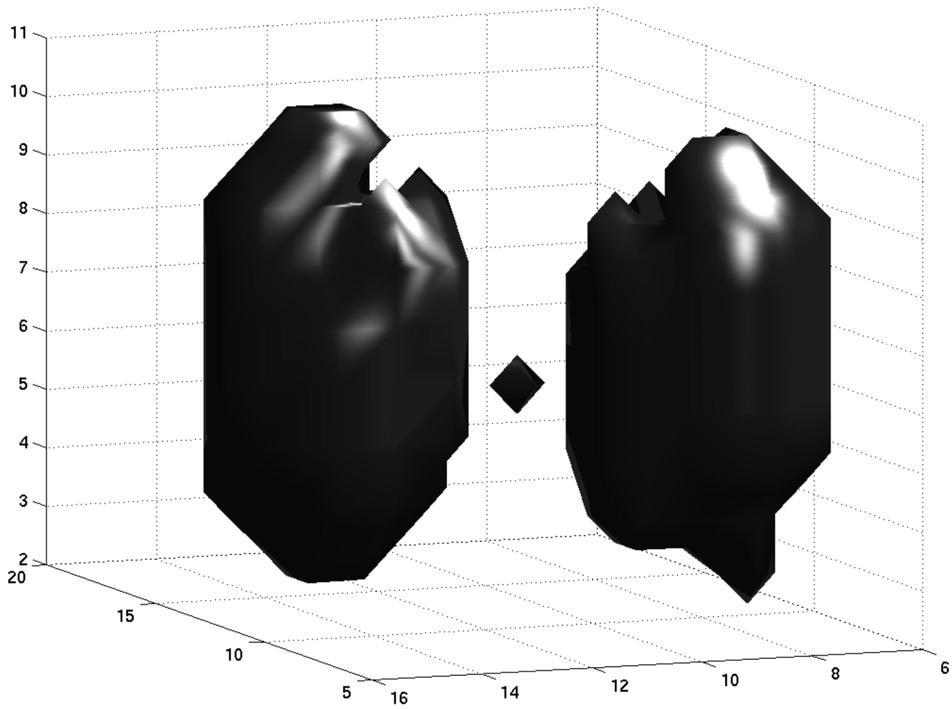


Initial Position Estimate
True Position
Final Position Estimate
z (mm.)
15
14
13
12
11
10
9
10
12
14
16
18
x (mm.)
10
15
20
y (mm.)

FIG. 1:



(a) The actual shape of the inhomogeneity



(b) The final shape estimate.

FIG. 2: