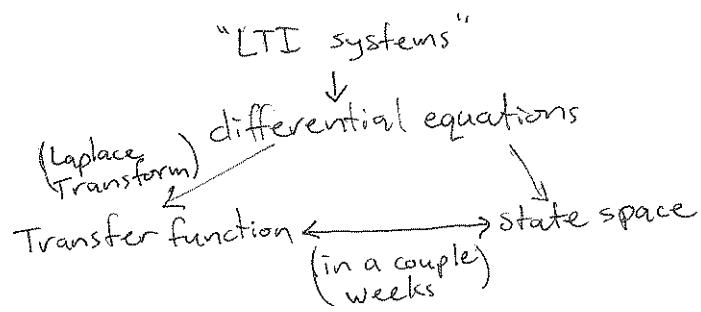
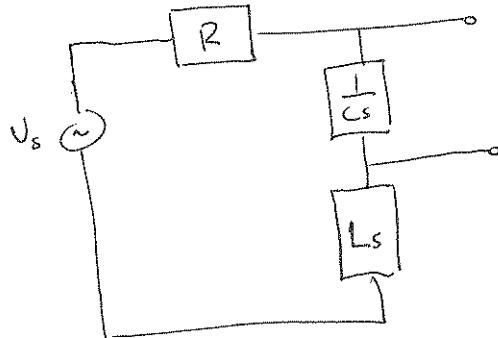
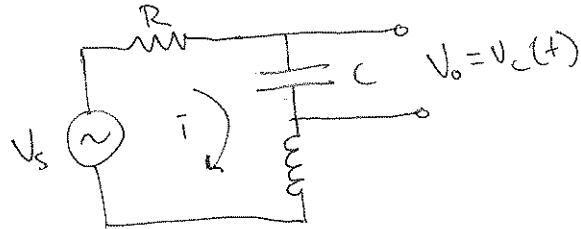


Where we are:



Let's start with an example:



$$V_s = Ri + V_C + L\dot{I} \quad (\text{By KVL})$$

$$i = C\dot{V}_C \quad (\text{aka } C \frac{dV}{dt})$$

$$V_s = RC\ddot{V}_C + V_C + LC\ddot{V}_C$$

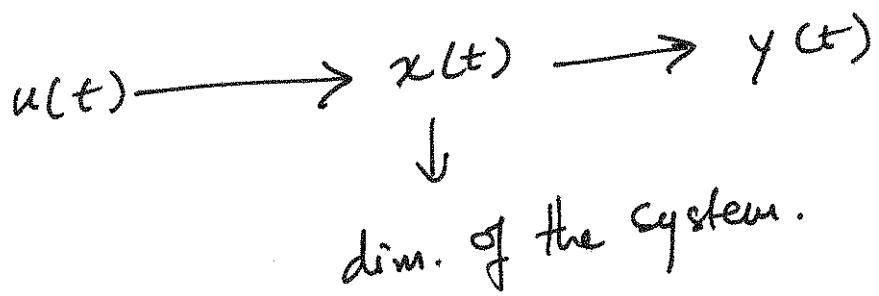
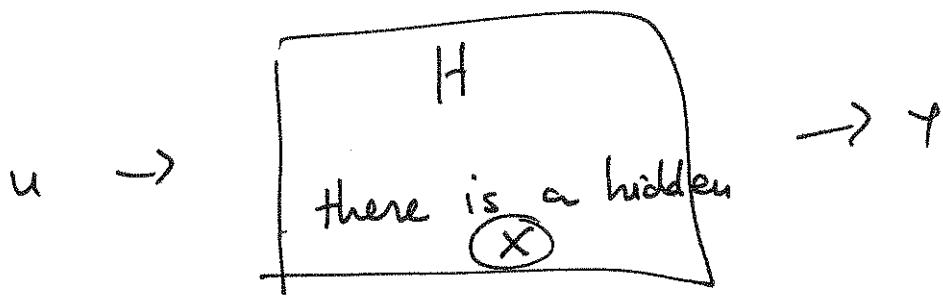
$$V_s = LC\ddot{V}_C + RC\dot{V}_C + V_C \quad (\text{reorder})$$

$$V(s) = (LCs^2 + RCS + 1)Y(s) \quad (\text{Laplace transform})$$

$$\frac{Y(s)}{V(s)} = \frac{1}{LCs^2 + RCS + 1} = H(s) \quad (\text{Transfer function})$$

We can also write this as a set of first-order linear differential equations.
And then put them in a matrix!

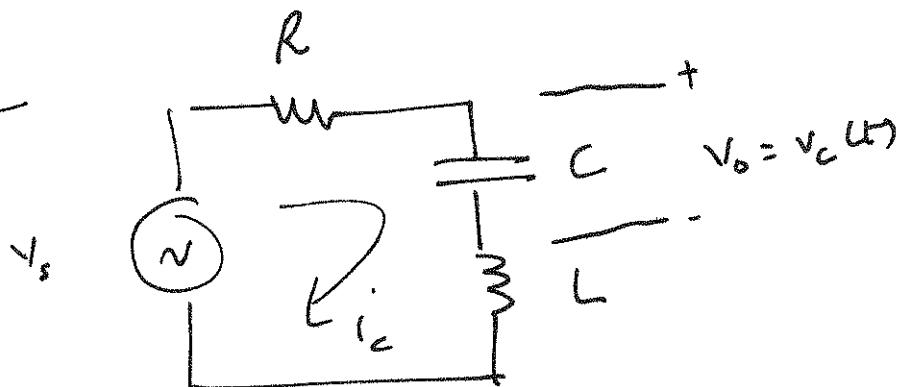
(8)



How do we characterize x ?

$x(t)$ is called state space vector
 $y(t)$ is output.
 $u(t)$ is input.

(9)

Example

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$v_s(t) = v_R(t) + v_c(t) + v_L(t)$$

$$= i_c^o R + v_c + L i_c^o$$

$$i_c^o = -\frac{1}{L} v_c - \frac{R}{L} i_c^o + \frac{1}{L} v_s$$

$$i_c^o = \frac{1}{C} i_c$$

go back to
the start
question

$$\begin{bmatrix} v_c \\ i_c \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -R/L \end{bmatrix} \begin{bmatrix} v_c \\ i_c \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_s$$

$$\dot{x} = Ax + Bu \quad \left\{ \begin{array}{l} y = [1 \ 0] x \\ \end{array} \right.$$

Is a state-space matrix unique? No!
 If we pick different variables, we get a different matrix.

Go back to "standard form" of diff. eq:

$$V_s = LC\ddot{V}_c + RC\dot{V}_c + V_c \Rightarrow \ddot{V}_c = -\frac{R}{L}V_c - \frac{1}{LC}V_c + \frac{1}{LC}V_s$$

A standard way to write the SS: (Ogata 2.5)

$$\text{Let } x_1 = V_c$$

$$x_2 = \dot{V}_c$$

Therefore,

$$\dot{x}_1 = x_2 \text{ (easy!)}$$

$$\dot{x}_2 = -\frac{R}{L}x_2 - \frac{1}{LC}x_1 + \frac{1}{LC}u$$

In matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u$$

Output V_c is just x_1 :

$$y = [1 \ 0]$$

LECTURE 2

①

RLC circuits are LTI?

RLC circuits are represented by Diff.
eqs.

A system of ^{linear} DEQ's with constant coeff. represent LTI systems.

$$\overset{\circ}{x}_1 = ax_1 + bu$$

① $\overset{\circ}{x}_2 = cx_2 + du$

$$\overset{\circ}{x}_1 = ax_2 + bu$$

② $\overset{\circ}{x}_2 = ax_1 + du$

Both ① and ② can be written
in a matrix form.

$$\textcircled{1} \quad \begin{bmatrix} \overset{\circ}{x_1} \\ x_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix} \textcircled{2} u$$

$$\textcircled{2} \quad \begin{bmatrix} \overset{\circ}{x_1} \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & a \\ c & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix} u$$

\textcircled{2} has another form:

What is it?

$$\ddot{x}_1 = a\overset{\circ}{x}_2 + \overset{\circ}{b}u$$

$$\ddot{x}_1 = a(cx_1 + du) + \overset{\circ}{b}u$$

Something more complicated.

(3)

$$\ddot{x}_1 = a_1 \dot{x}_1 + a_2 \dot{x}_2 + b_1 u$$

$$\ddot{x}_2 = a_3 \dot{x}_1 + a_4 \dot{x}_2 + b_2 u$$

$$\underline{\underline{[x_1] = [a_1 \ a_2] [x_1] + [b_1] u}}$$

$$\ddot{x}_1 = a_1 \dot{x}_1 + a_2 \dot{x}_2 + b_1 u$$

$$= a_1 \dot{x}_1 + a_2 (a_3 x_1 + a_4 x_2) + b_2 u + b_1 u$$

$$= a_1 \dot{x}_1 + a_2 a_3 x_1 + a_2 a_4 x_2 + a_2 b_2 u + b_1 u$$

$$\frac{\dot{x}_1 - a_1 x_1 - b_1 u}{a_2} = x_2$$

$$= a_1 \dot{x}_1 + a_2 a_3 x_1 + a_2 a_4 \left(\frac{\dot{x}_1 - a_1 x_1 - b_1 u}{a_2} \right) + a_2 b_2 u + b_1 u$$

$$= a_1 \dot{x}_1 + a_4 \dot{x}_1 + a_2 a_3 x_1 - \frac{a_4}{a_1} a_1 x_1 - a_4 b_1 u + a_2 b_2 u + b_1 u$$

$$\ddot{x}_1 = (a_1 + a_4) \dot{x}_1 + (a_2 a_3 - a_4 a_1) x_1 + (a_2 b_2 - a_4 b_1) u + b_1 u$$

$$\underbrace{\ddot{x}_1 = \tilde{a}_1 \ddot{x}_1 + \tilde{a}_2 x_1 + \tilde{b} u + b \dot{u}}$$

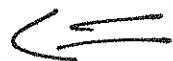
(4)

Write this in matrix form.

$$\begin{bmatrix} \ddot{x}_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \tilde{a}_2 & \tilde{a}_1 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} + \begin{bmatrix} \cancel{-b} \\ \cancel{b} \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \end{bmatrix}$$

SUMMARY

LTI systems \Rightarrow Diff. eq.



All LTI systems can be written
in a matrix form.

In general,

$$\ddot{x} = Ax + Bu$$

State space with derivatives of input.

We have the equation

$$\ddot{y} + a_1\dot{y} + a_2y = b_1\dot{u} + b_2u$$

$$\ddot{y} = -a_1\dot{y} - a_2y + b_1\dot{u} + b_2u$$

We want to put this into the form

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = A \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + B u$$

$$y = Cx$$

Need 2nd derivative of y , only 1st derivative of u

$$\left. \begin{array}{l} \text{Let } x_1 = y \Rightarrow \dot{x}_1 = \dot{y} \\ x_2 = \dot{x}_1 - b_1u = \dot{y} - b_1u \Rightarrow \dot{x}_2 = \ddot{y} - b_1\dot{u} \end{array} \right\} \begin{array}{l} \text{define } x_1/x_2 \text{ in terms} \\ \text{of } y/g/u \end{array}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A & \\ & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ \end{bmatrix} u$$

Now get the coefficients for A and B by finding equations

for \dot{x}_1 and \dot{x}_2 in terms of x_1 , x_2 , and u .

$$\dot{x}_1 = x_2 + b_1u \quad (\text{by definition of } x_2)$$

$$\dot{x}_2 = \ddot{y} - b_1u = -a_1(\underbrace{\dot{y}}_{x_1}) - a_2y + b_2u \quad \begin{array}{l} (\text{Using definition of } x_2 \text{ and original} \\ \text{eqn}) \end{array}$$

want this in terms
of x_1 and x_2

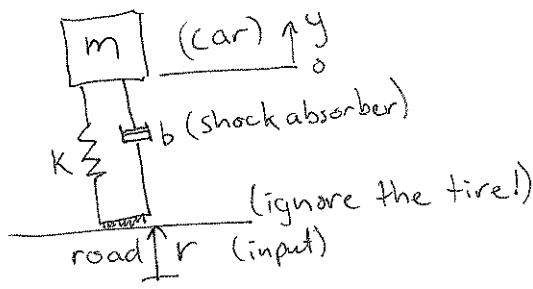
$$\begin{aligned} \dot{x}_2 &= -a_2x_1 + (-a_1\dot{y} + b_2u) = -a_2x_1 - a_1\dot{y} + a_1b_1u - \underbrace{a_1b_1u}_{\text{Add + subtract}} + b_2u \\ &= -a_2x_1 + -a_1(\dot{y} - b_1u) + (b_2 - a_1b_1)u \\ &= -a_2x_1 + -a_1x_2 + (b_2 - a_1b_1)u \end{aligned}$$

Finally, in matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 - a_1b_1 \end{bmatrix} u$$

$$y = [1 \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Another example: car suspension (simplified from FPE Ex. 2.2)



Designate y to be upward displacement.

$$F = m \cdot a = m \cdot \ddot{y} = -k(y - r) - b(\dot{y} - \dot{r})$$

distributing terms,

$$m \ddot{y} = -ky - b\dot{y} + kr + b\dot{r}$$

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{r} + \frac{k}{m}r$$

Laplace transform

$$(s^2 + \frac{b}{m}s + \frac{k}{m})Y(s) = (\frac{b}{m}s + \frac{k}{m})R(s)$$

$$H(s) = \frac{Y(s)}{R(s)} = \frac{\frac{b}{m}s + \frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \quad \leftarrow \text{Transfer function}$$

To state space:

$$\text{let } x_1 = y$$

$$x_2 = \dot{y} - \frac{b}{m}r = \dot{x}_1 - \frac{b}{m}r$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{b}{m}r \\ \frac{k}{m}r \end{bmatrix}$$

$$y = [1 \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Just to check, eigenvalues of A are

$$|A - \lambda I|$$

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{b}{m} - \lambda \end{vmatrix} = \lambda\left(\frac{b}{m} + \lambda\right) + \frac{k}{m}$$

$$= \lambda^2 + \frac{b}{m}\lambda + \frac{k}{m}$$

Roots of this equation match poles of TF.