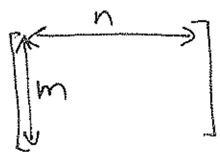
Lecture 3 9/11/19

Today: linear algebra, next Monday: applying these to state space

What is a matrix?

$A =$ 

we say $A \in \mathbb{R}^{m \times n}$

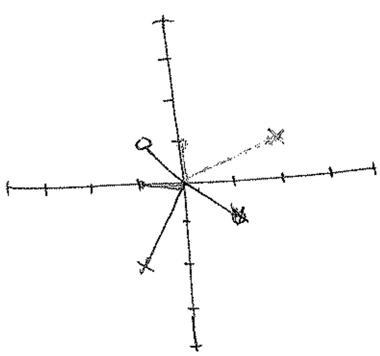
What does a matrix do to a vector?

$y = Ax \quad (x \in \mathbb{R}^n)$

Produces $y \in \mathbb{R}^m$ – a linear transformation between spaces

This is most interesting for high-dimensional matrices, but let's stick with 2x2

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad x = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

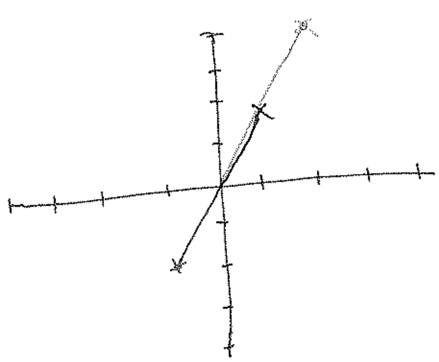


What does this mean in state space?

Matrix A takes a state and gives a new state in $\mathbb{R}^n$.

What about

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$? $\quad x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad x = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



Everything gets mapped onto a line! Why?

y is a linear combination of columns, which are not linearly independent.

Rank

3 cases:

$y = Ax$ is always 0 — rank $= 0$

$y = Ax$ is a subspace of $\mathbb{R}^n$ — $0 < \text{rank} < n$

$y = Ax$ is anywhere in $\mathbb{R}^n$ — rank $= n$ ("full rank")

Nullspace

When A is full rank, only $x = 0$ can give $Ax = 0$

But if A is rank-deficient, there are many x which give $Ax = 0$

Scaling in particular:

$$A(\alpha x) = 0$$
$$\alpha A x = 0$$
$$A x = 0$$

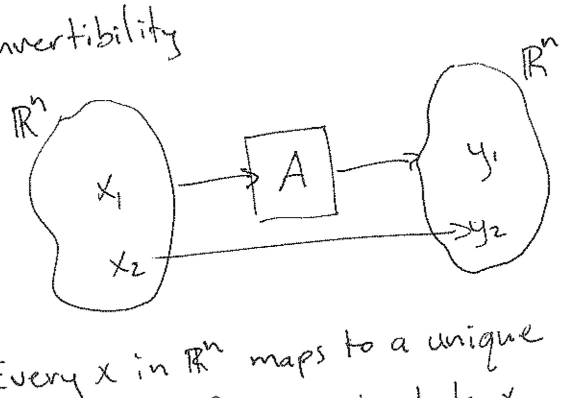
pick x such that $Ax = 0$, then any $\alpha \in \mathbb{R}$

This is the nullspace of A.

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ let $x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(For a square, symmetric matrix) nullspace is orthogonal to range(A). More generally, nullspace(A^T) $\perp$ range(A)

Invertibility



Every x in $\mathbb{R}^n$ maps to a unique y in $\mathbb{R}^n$ iff A is full rank

↳ can go from y back to x

A^{-1} exists.

Eigenvalues/eigenvectors

Some vector x such that

$$Ax = \lambda x$$

What does this mean?

A only scales x, doesn't rotate it

If the state is x, will always be αx (if the system is autonomous)

But eigenvalues/eigenvectors also help us think about <u>all</u> x.

$$A \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Right-multiply by V^{-1} to get

$$A = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

$A = VDV^{-1}$ (But only if V^{-1} exists)

Breaks down A into 1) rotate (V^{-1})

2) scale along axes (D or Λ)

3) rotate again (V)

Eigenvectors are the scaling axes

Eigenvalues are the scaling amount

↳ write a vector as a linear combination of eigenvectors,

now it's easy to think about what happens to it.

Exploration: What do the eigenvalues tell us about the behavior of the system?

Comments on Lecture 2

Fall 2013 - EE 105, Feedback Control Systems (Prof. Khan)
September 09, 2013

I. EIGENVECTORS AND EIGENVALUES

Choose a vector, $\mathrm{x} \in \mathbb{R}^2$, such that x lies in the unit circle, i.e., with length less than or equal to one. Mathematically, the length of a vector is denoted by $\|\mathrm{x}\|_2$, defined as

$$\|\mathrm{x}\|_2 = \sqrt{x_1^2 + x_2^2}, \tag{1}$$

where x_1 and x_2 are the horizontal and vertical components of x. A plot of $25,000$ such x's is shown in Fig. 1.

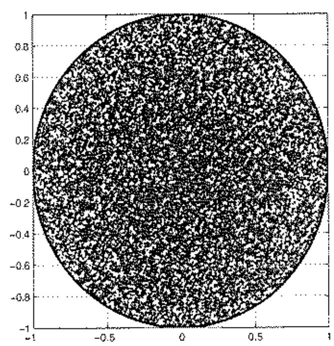


Fig. 1. Points generated arbitrarily inside the unit circle.

Choose a matrix, $A \in \mathbb{R}^{2\times 2}$, such that A is symmetric (the reason to be apparent in Lecture 3) and its two-norm, denoted as $\|A\|_2$, is 1. The two-norm of a symmetric matrix is defined as

$$\|A\|_2 \triangleq \sqrt{A^T A} = \max_{i=1,2} \lambda_i, \tag{2}$$

where λ_i is the ith eigenvalue of A.

Lets consider the following operation,

$$\mathrm{y} = A\mathrm{x}, \tag{3}$$

over all possible values of x in the unit circle. In other words, we do not constrain the collection of x's to have a particular structure other than a bound on its length. With this length constraint, we can have a handle on the length of $\mathbf{y}$:

$$
\begin{aligned}
||\mathbf{y}||_2 &= ||A\mathbf{x}||_2, \\
&\leq ||A||_2||\mathbf{x}||_2, \qquad \text{Sub-multiplicative property of two-norm} \\
&= 1.
\end{aligned}
$$

In other words, we know that for any $\mathbf{x}$ within the unit circle and for any matrix, A, with two-norm of 1, the operation, $A\mathbf{x}$, will return a vector, $\mathbf{y}$, whose length, $||\mathbf{y}||_2$, will be no larger than 1. For the $\mathbf{x}$'s shown in Fig. 1, the corresponding $A\mathbf{x}$'s are shown in Figs. 2. The corresponding A matrix is also shown.

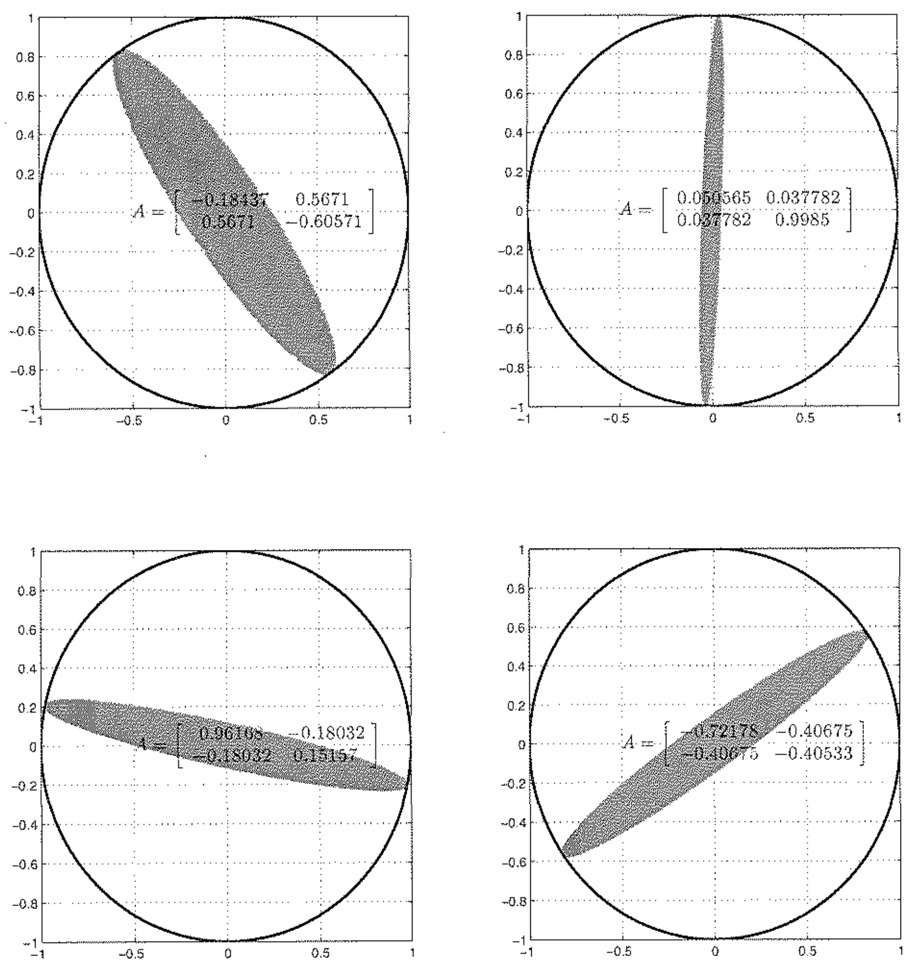


Fig. 2. Different matrices operating on the points within the unit circle.

Here is the key question: Why is it that x's were only constrained to have at most unit length but the resulting Ax's lie in some ellipse (in addition to being at most unit length)? Furthermore, each ellipse is different for a different matrix, A. Intuitively, a particular matrix, A, must have something to do with the resulting ellipse. *Convince yourself that an ellipse is essentially a generalized circle.*

We understand that a matrix (real-valued) rotates and scales a vector. But in addition, this rotation and scaling are also different in different directions. Eigenvectors define such directions along which the matrix would rotate a vector, and eigenvalues define the corresponding scaling.

As an example, consider the bottom right figure in Fig. 2, where

$$A = \begin{bmatrix} -0.7218 & -0.4068 \\ -0.4068 & -0.4053 \end{bmatrix}. \tag{4}$$

The eigenvectors and eigenvalues of $A = VDV^{-1}$ are

$$V = \begin{bmatrix} -0.8254 & 0.5646 \\ -0.5646 & -0.8254 \end{bmatrix}, \qquad D = \begin{bmatrix} -1.0000 & 0 \\ 0 & -0.1271 \end{bmatrix}. \tag{5}$$

In Fig. 3, the blue lines are the eigenvectors *multiplied* by the corresponding eigenvalues. This multiplication is necessary as the eigenvectors, by definition, are of unit length. Without this multiplication, both blue lines will touch the unit circle. However, the contribution to the scaling in each eigenvector's direction is scaled by the corresponding eigenvalue.

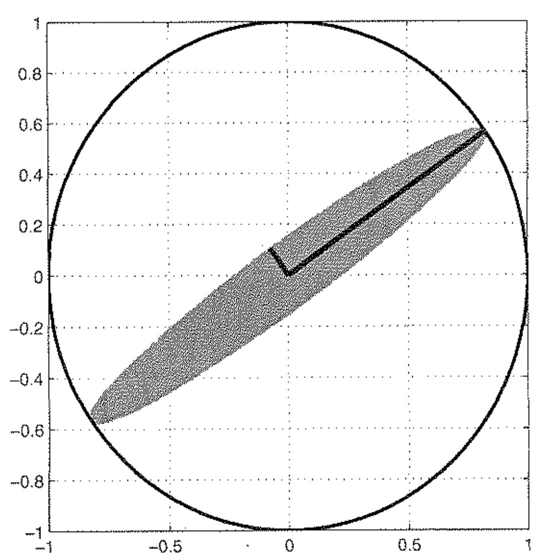


Fig. 3. Eigenvalues and eigenvectors

Try the following code with different A matrixes. Specifically, consider rank-deficient matrices and make peace with the outcome. Below is the code to generate Figs. 1 and 2.

```
clear
clc
close all

A = randn(2);
A = A + A';
A = A/norm(A);

X = [];
Y = [];

%%% Generate 25000 points for x within unit circle
while length(X) < 25000
    x = randn(2,1);
    if norm(x) <=1
        X = [X x];
        Y = [Y A*x];
    end
end

plot(X(1,:),X(2,:), '.')
hold on
plot(real(exp(2*pi*i*[0:100]/100)),imag(exp(2*pi*i*[0:100]/100)), 'linew', 2, 'color', 'black')
axis([-1 1 -1 1])
grid on
set(gcf, 'PaperPosition', [0 0 5 5]); %Position plot at left hand corner with width 5 and height
set(gcf, 'PaperSize', [5 5]); %Set the paper to have width 5 and height 5.
box on
saveas(gca,'ev_fig1','pdf')


figure
plot(Y(1,:),Y(2,:), 'r.')
hold on
plot(real(exp(2*pi*i*[0:100]/100)),imag(exp(2*pi*i*[0:100]/100)), 'linew', 2, 'color', 'black')
text(-.5,0, ['$A=\left[\begin{array}{cc}' num2str(A(1,1)) '&' num2str(A(1,2)) '\\' ...
    num2str(A(2,1)) '&' num2str(A(2,2)) '\end{array}\right]$'], 'interp', 'latex', 'fonts', 14)
axis([-1 1 -1 1])
grid on
set(gcf, 'PaperPosition', [0 0 5 5]); %Position plot at left hand corner with width 5 and height
set(gcf, 'PaperSize', [5 5]); %Set the paper to have width 5 and height 5.
box on
saveas(gca,'ev_fig2','pdf')
```

Below is the code to generate Fig. 3.

```
clear
clc
close all

A = [    -0.7218    -0.4068;    -0.4068    -0.4053];
[V D] = eig(A);

X = [];
Y = [];

%%% Generate 25000 points for x within unit circle
while length(X) < 25000
    x = randn(2,1);
    if norm(x) <=1
        X = [X x];
        Y = [Y A*x];
    end
end

figure
plot(Y(1,:),Y(2,:), 'r.')
hold on
plot(real(exp(2*pi*i*[0:100]/100)),imag(exp(2*pi*i*[0:100]/100)), 'linew', 2, 'color', 'black')
axis([-1 1 -1 1])
grid on
plot([0 D(1,1)*V(1,1)],[0 D(1,1)*V(2,1)], 'linew', 3)
plot([0 D(2,2)*V(1,2)],[0 D(2,2)*V(2,2)], 'linew', 3)

set(gcf, 'PaperPosition', [0 0 5 5]); %Position plot at left hand corner with width 5 and height
set(gcf, 'PaperSize', [5 5]); %Set the paper to have width 5 and height 5.
box on
saveas(gca,'ev_fig3','pdf')
```