

Leftovers from lecture 4

If we have a matrix A , which has an eigen-decomposition $A = VDU^{-1}$
 The natural response (i.e., no input) will be:

$$\begin{aligned} y(t) &= C \cdot x(t) \\ &= C \cdot e^{At} x(0) \\ &= C \cdot V \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \end{bmatrix} V^{-1} x(0) \end{aligned}$$

We examined the eigenvalues (λ_n) to determine stability.

If $\lambda < 0$, $x(t) \Rightarrow 0$ as $t \rightarrow \infty$ and is stable

If $\lambda > 0$, $x(t) \rightarrow \infty$ as $t \rightarrow \infty$ which is unstable.

What about complex λ ?

$$\begin{aligned} e^{(\alpha + j\beta)t} &= e^{\alpha t} e^{j\beta t} \\ &= e^{\alpha t} \underbrace{(\cos(\beta t) + j \sin(\beta t))}_{\substack{\text{Magnitude of 1} \\ \text{Depends on } \alpha, \\ \text{as before.}}} \end{aligned}$$

Intuition for real part - if $t=0$, $e^{j \cdot 0} = 1$, which is real.

Does this mean we get a complex bit in our output?

No. Eigenvalues always come in complex-conjugate pairs, which cancel out.

$$(\lambda^2 + 1) = 0 \Rightarrow (\lambda + j)(\lambda - j) = 0$$

Intuitively, the exponential of a real matrix must be real.

Even for fractional powers? Yes, remember the Taylor expansion.

[Show this in MATLAB] rlc.m

What about stability under input?

"Forced response" or "External response"

External response
Forced response
Particular solution

when $u(t) \neq 0$
 and $x(0) = 0$

Try to find $x(t) = \dots$ with A and $Bu(t)$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$e^{-At} \dot{x}(t) = e^{-At} A x(t) + e^{-At} Bu(t)$$

product rule \rightarrow

$$e^{-At} \dot{x}(t) - e^{-At} A x(t) = e^{-At} Bu(t)$$

$$\frac{d}{dt} (e^{-At} x(t)) = e^{-At} Bu(t)$$

$$\int_{t_0}^t \frac{d}{d\tau} (e^{-A\tau} x(\tau)) d\tau = \int_{t_0}^t e^{-A\tau} Bu(\tau) d\tau$$

with respect to limit

limit to t_0 = initial condition = 0

$$\frac{e^{-At} x(t)}{e^{-At}} = \frac{e^{-A(t_0)} x(t_0)}{e^{-A(t_0)}} = \int_{t_0}^t e^{-A\tau} Bu(\tau) d\tau$$

$$x(t) = e^{At} \int_{t_0}^t e^{-A\tau} Bu(\tau) d\tau$$

$$= \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau$$

Total response is Internal + External

(3)

$$y(t) = \underbrace{C e^{At} x(0)}_{\text{internal}} + C \int_0^t e^{A(t-\tau)} B \cdot u(\tau) d\tau \quad (\text{ignoring } D)$$

Is this stable?

Convergence to 0 doesn't help much... $u(t)$ could be anything and go on forever.

BIBO: given a finite input, is the output bounded?

Only if integral of impulse response is finite.

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

What is the impulse response for a state-space formulation?

$u(t)$ is an impulse at $t=0$, $x(0) = 0$

$$y(t) = C e^{At} \cdot B, \text{ designated } h(t)$$

This is a perfect segue...

$h(t)$ is the impulse response, so if we took the Laplace transform, we'd have the transfer function.

$\mathcal{L}\{e^t\} = \frac{1}{s-1}$, so $\mathcal{L}\{e^{At}\}$ with a matrix A is $(sI - A)^{-1}$

$$H(s) = C(sI - A)^{-1}B$$

We could write this out as

$$H(s) = \frac{\text{some polynomial of } s}{|sI - A|}$$

\ determinant

because $P^{-1} = \frac{1}{|P|} \text{adj}(P)$

Remember how we calculate eigenvalues?

$|\lambda I - A|$, find the roots

Roots of $|sI - A|$ are the poles. which are the eigenvalues.