

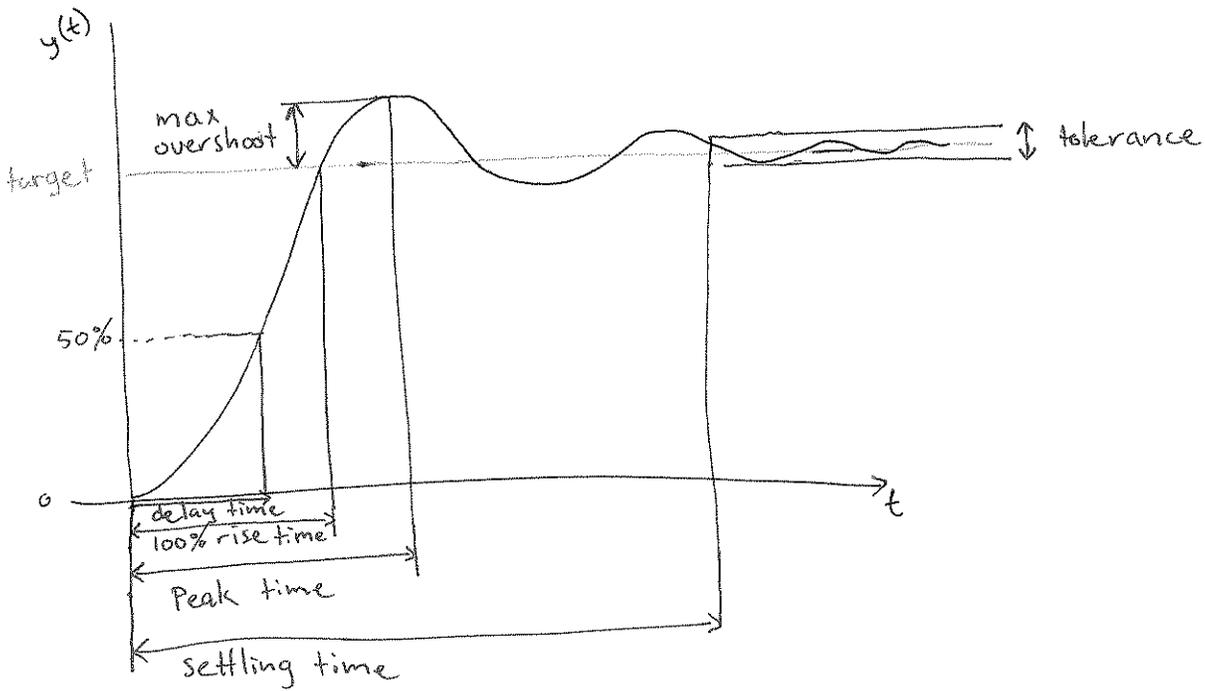
# Lecture 7

①

We defined some desirable characteristics back at the beginning of the course

- stable (we have tools for this already)
- speed/time to target — Delay time, rise time, peak time, settling time
- output matches target — Tolerance
- handle (abrupt) disturbances
- keeps output steady — Maximum overshoot

Let's try to quantify these:



How do these relate to poles of the system?

Let's start with a 2nd-order system

$$H(s) = \frac{k}{s^2 + bs + c}$$

Focus on finding the poles - roots of the denominator

We're going to make up two new parameters:  $\omega_n$  and  $\zeta$

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\omega_n = \sqrt{c}$$

$$\zeta = \frac{b}{2\omega_n}$$

Using the quadratic formula,

$$\frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}$$
$$= -\zeta\omega_n \pm j\sqrt{1 - \zeta^2}$$

So we could write,

$$(s + \zeta\omega_n + j\sqrt{1 - \zeta^2})(s + \zeta\omega_n - j\sqrt{1 - \zeta^2})$$

which doesn't look like an improvement.

$$\frac{1}{s^2 + bs + c}$$

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

$$s^2 + 2bs + c$$

$$\frac{-2b \pm \sqrt{4b^2 - 4c}}{2} = -b \pm \sqrt{b^2 - c}$$

$$(s + \sigma)^2 + \omega_d^2 = s^2 + 2\sigma s + \sigma^2 + \omega_d^2$$

$$\zeta \quad \zeta \quad \zeta \quad \zeta \quad \zeta \quad \zeta \quad \zeta$$

- I took Greek in grad school, so you'd think I'd be better at this.

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\omega_n = \sqrt{c}$$

$$\zeta = \frac{b}{2\omega_n}$$

quadratic formula

$$-\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2}$$

4's drop out

$$-\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

$$-\zeta\omega_n \pm \omega_n j \sqrt{1 - \zeta^2}$$

Now we can write

$$(s + \zeta\omega_n \pm \omega_n j \sqrt{1 - \zeta^2})$$

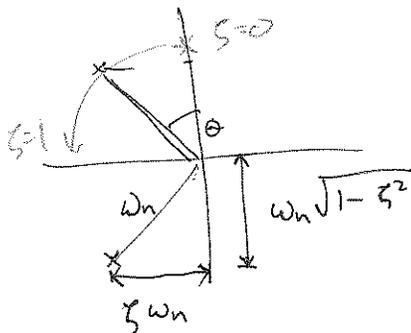
questionable improvement.

But wait!

Poles are  $\underbrace{\zeta\omega_n}_{\text{real}} \pm j \underbrace{\omega_n \sqrt{1 - \zeta^2}}_{\text{imag}}$

This gives me a way to think about poles, i.e., pick  $\zeta$  and  $\omega_n$ , then solve for parameters.

Can we have  $\zeta > 1$ ? Sure, but then the roots are real and this is silly.



diagonal is  $\sqrt{(\omega_n \sqrt{1 - \zeta^2})^2 + (\zeta\omega_n)^2} = \omega_n$

$$\theta = \frac{\zeta\omega_n}{\omega_n} = \zeta$$

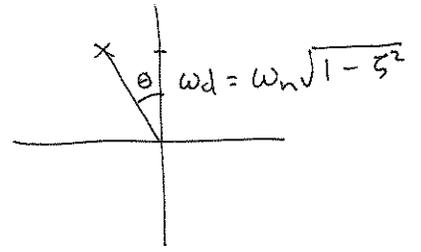
$\omega_n$  is natural frequency - oscillation frequency if no damping.

Do an example!

Back to our metrics:

Empirically, rise time  $t_r \approx \frac{1.8}{\omega_n}$  (FPE) 0.1 to 0.9

$$= \frac{\pi}{2} \cdot \theta \quad (\text{Ogata 5-19}) \quad 0 \text{ to } 1$$



Key point: for small  $t_r$ ,  $\omega_n$  must be large.

Peak time occurs when  $y(t)$  is at its maximum  $\Rightarrow \dot{y}(t) = 0$ .

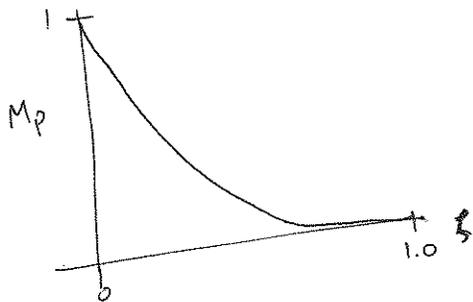
$$t_p = \frac{\pi}{\omega_d}$$

Overshoot  $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$  for  $0 \leq \zeta < 1$

This is for 0-1, multiply by 100 for percentage.

This is gross, let's just plot it. (FPE 3.24)

common values:  $\zeta = 0.5 \Rightarrow M_p = 0.16$  (16%)  
 $\zeta = 0.7 \Rightarrow M_p = 0.05$  (5%)



Tolerance/settling time

Pick a tolerance, say 1% = 0.01.

Decaying exponential should drop below this

$$e^{-\omega_n \zeta t} = 0.01$$

$$\omega_n \zeta t = 4.6$$

$$t = \frac{4.6}{\omega_n \zeta}$$