EE 105 Feedback control systems

Matrix exponential + time response

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By the end of class today, you should be able to:

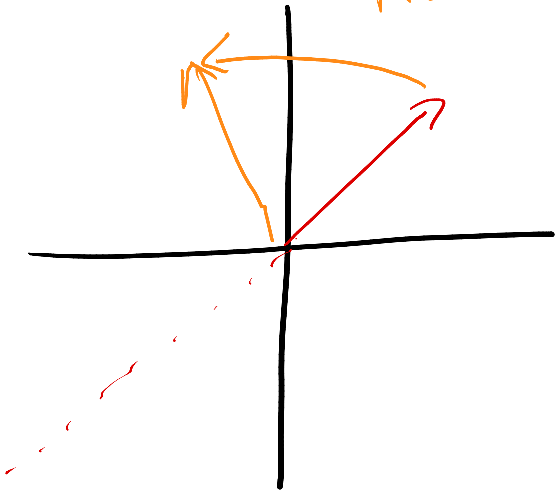
- Use MATLAB to compute a matrix exponential, and use this to plot the time-domain response

- Given a matrix representing an LTI system, determine whether it is stable

Rewind to lecture 3

What about matrices that don't have eigenvalues/eigenvectors?

Rotation matrix?



Rewind to lecture 3

What about matrices that don't have eigenvalues/eigenvectors?

$$|A - \lambda I|$$

$$= \begin{vmatrix} a_{00} - \lambda & a_{01} \\ a_{10} & a_{11} - \lambda \end{vmatrix}$$

$$= \left((a_{00} - \lambda)(a_{11} - \lambda) - a_{10}\, a_{01}\right)$$

eigenvalues are roots of this polynomial

From dynamics matrix to output

We have $\dot{x} = Ax$

$$y = Cx$$

x is e.g. $\begin{bmatrix} \text{voltage} \\ \text{current} \end{bmatrix}$

$C = [1, 0]$

and we want to find the output as a function of time

For scalars,

$\dot{x} = \alpha x \qquad \left(\frac{dx}{dt} = \alpha x \right)$

$x(t) = e^{\alpha t} \cdot x(0)$

For matrices

$$\begin{bmatrix} \dot{x} \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} x \end{bmatrix}$$

should give

$$x(t) = \boxed{e^{At}} \cdot x(0)$$

?

A, A^2, A^3, etc exist

All we need is e^{At}

Power series!

Let

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$\frac{d}{dt} e^{At} = \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \right)$$

$$= 0 + A + \frac{A^2\, 2t}{2!} + \frac{A^3\, 3t^2}{3!} + \cdots$$

$$= A\left(I + At + \frac{A^2t^2}{2!} + \cdots \right)$$

Properties of matrix exponential

Can we take a derivative?

A more intuitive way

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} + \begin{bmatrix} a_{11}t & & & \\ & a_{22}t & & \\ & & \ddots & \\ & & & a_{nn}t \end{bmatrix} + \begin{bmatrix} \frac{a_{11}^2t^2}{2} & & & \\ & \frac{a_{22}^2t^2}{2} & & \\ & & \ddots & \\ & & & \frac{a_{nn}^2t^2}{2} \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + a_{11}t + \frac{a_{11}t^2}{2!} + \cdots & & \\ & 1 + a_{22}t + \frac{a_{22}^2t^2}{2!} & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} e^{a_{11}t} & & \\ & e^{a_{22}t} & \\ & & \ddots \end{bmatrix}$$

A more intuitive way

Assume $A = VDV^{-1}$ exists

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$A^2 = VDV^{-1}VDV^{-1}$$
$$= VD^2V^{-1}$$

$$= VD^0V^{-1} + VDV^{-1}t + \frac{1}{2!}VD^2V^{-1}t^2 + \frac{1}{3!}VD^3V^{-1}t^3 + \cdots$$

$$= V\left(I + D + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!} + \cdots\right)V^{-1}$$

$$= Ve^{Dt}V^{-1}, \quad \text{or} \quad V\begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}V^{-1}$$

The temporal response

no input, "autonomous system"

$$y(t) = C\,x(t)$$

$$= C\,e^{At}\,\underline{x(0)} \quad \text{initial condition}$$

$$= C\,V \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \end{bmatrix} V^{-1}\,x(0)$$

The temporal response

What about complex X?

$$e^{(a+bj)t} = e^{at} e^{jbt}$$

$$= \underbrace{e^{at}}_{\text{depends on } a} \underbrace{\left(\cos(bt) + j\sin(bt)\right)}_{\text{magnitude of } 1}$$

The temporal response

$$y(t) = \underbrace{C e^{At} x(0)}_{\text{internal}} + C \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$\begin{bmatrix} \dot{x} \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} B \end{bmatrix} u(t)$$