Following the processing chain in the receiver:

\[ x(t) = s(t) + n(t) \]
\[ = A_c \cos(2\pi f_c t) \bar{m}(t) + z_1(t) \cos(2\pi f_c t) - z_2(t) \sin(2\pi f_c t) \]

\[ \nu(t) = x(t) \cos(2\pi f_c t) \]
\[ = \frac{1}{2} A_c \bar{m}(t) + \frac{1}{2} z_1(t) \]
\[ + \frac{1}{2} \left( A_c \bar{m}(t) + z_1(t) \right) \cos(2\pi f_c t) - \frac{1}{2} A_c z_2(t) \sin(2\pi f_c t) \]

After LPF, then we obtain

\[ y(t) = \frac{1}{2} A_c \bar{m}(t) + \frac{1}{2} z_1(t) \]

Only the in-phase noise component left!

The signal power in \( y(t) \) is \( \frac{A_c^2}{4} P \).

The noise power in \( y(t) \) is \( \left( \frac{1}{2} \right) \cdot 2 N_0 W = \frac{N_0 W}{2} \),

because the bandwidth of the BPF is \( 2W \).

Thus the output SNR is

\[ (\text{SNR})_o = \frac{\frac{1}{2} A_c^2 P}{\frac{1}{2} W N_0} = \frac{A_c^2 P}{2 N_0 W} \]

Then the figure-of-merit for DSB-SC systems is:

\[ \frac{(\text{SNR})_o}{(\text{SNR})_c \text{ DSB-SC}} = 1 \]

- Noise in AM receiver:

Consider the full AM signal that can use an envelope detector in the receiver:

\[ s(t) = A_c \left[ 1 + k a(m(t)) \right] \cos(2\pi f_c t) \]
- Channel SNR:
  1. Average power of the carrier component = $A_c^2/2$
  2. Average power of the information-bearing component = $\frac{A_c^2P}{2}$
  3. Average power of noise within the message bandwidth $N_0W$

  Thus, $(SNR)_c = \frac{1}{2} \frac{A_c^2 + A_c^2P}{N_0W}$

- Output SNR: Again follow the Rx processing chain:
  1. Filtered signal
     \[ x(t) = s(t) + n(t) \]
     \[ = \left[ A_c + A_c k_a m(t) + n_I(t) \right] \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \]
  2. Output signal $y(t)$ is the envelope of $x(t)$.

  **Phasor diagram**
  
  Since the in-phase and quadrature components are orthogonal, the envelope of $x(t)$ is:

  \[ y(t) = \left( \left( A_c + A_c k_a m(t) + n_I(t) \right)^2 + n_Q(t)^2 \right)^{1/2} \]

  Assume that the carrier power is much higher than the noise power on average, which is often the case, we can,
simplify the expression for \( y(t) \) as
\[
y(t) \approx A_c + A_c k_a m(t) + \nu_E(t)
\]
The DC component \( A_c \) can be removed by a blocking capacitor (it bears no relation to \( m(t) \)).

Thus the output signal power is \( A_c^2 k_a^2 P \)
and the output noise power is approximately \( 2N_0 W \) (because the output after bandpass as bandwidth \( 2W \)).

\[
\rightarrow (SNR)_0 \approx \frac{A_c^2 k_a^2 P}{2N_0 W}
\]

Note that \((SNR)_0\) above is approximate under 2 conditions:
- average noise power is small compared to carrier power
- amplitude sensitivity \( k_a \) is such that percentage modulation is \( \leq 100\% \).

Then the figure of merit for AM systems is
\[
\frac{(SNR)_0}{(SNR)_c \frac{1}{AM}} = \frac{k_a^2 P}{1 + k_a^2 P} < 1
\]
(AM wastes power in transmitting the carrier.)

Threshold effect:
What if the noise power increases, or the carrier power decreases, such that the above approximation no longer holds?

Let's look at the case when the noise dominates the carrier. It is better in this case to express the noise in its polar form:
\[
n(t) = r(t) \cos (2\pi f_c t + \phi(t))
\]
\( \phi(t) \) is the phase, which is uniformly random between \( [-\pi, \pi] \).
Based on the above phasor diagram, the output is approx:
\[ y(t) = r(t) + A_c[1 + k_a m(t)] \cos \phi(t) \]
\[ = r(t) + A_c \cos \phi(t) + A_k a m(t) \cos \phi(t) \]

But we cannot really extract the message signal \( m(t) \) back from this \( y(t) \) output since the phase \( \phi(t) \) is unknown and random.

Thus at very low carrier-to-noise ratio (\( << 1 \)), there is a complete loss of information in AM system.

This effect is called the "threshold effect." That is, below a certain threshold on the carrier-to-noise ratio, the receiver performance deteriorates rapidly and information is lost.

Thus threshold effect is common in non-linear receivers such as the envelope detector.

It does not apply to the coherent detector.

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**Noise in FM receivers:**

- FM signals contain information in the phase.
- The amplitude of FM signals is constant. Thus any amplitude variation at the receiver input is due to noise (or interference).
- Thus in the FM receiver, the input signal is clipped and filtered before demodulation.
Noise after BPF:
\[ n(t) = n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \]
\[ = r(t) \cos \left[ 2\pi f_c t + \phi(t) \right] \]
where
- noise envelope \( r(t) = \left[ n_I^2(t) + n_Q^2(t) \right]^{1/2} \) Rayleigh distributed
- noise phase \( \phi(t) = \tan^{-1} \left( \frac{n_Q(t)}{n_I(t)} \right) \) uniform \([-\pi, \pi]\)

**FM Signal**
\[ s(t) = A_c \cos \left[ 2\pi f_c t + 2\pi f \int_0^t m(t) dt \right] \]
\[ = A_c \cos \left[ 2\pi f_c t + \phi(t) \right] \]

deleted nothing phase that contains information.

**Channel SNR:**
This is straightforward to compute.
- FM signal power \( = \frac{1}{2} A_c^2 \)
- Noise power on the message BW is \( N_0 W \)
\[ (SNR)_c = \frac{A_c^2}{2N_0 W} \]

**Output SNR:**
This part is much trickier to compute and requires some assumptions.
In particular, we will again assume that the carrier power is much larger than the noise power i.e., carrier-to-noise ratio \( > 71 \).
We are interested in the noise power at the output, after going through the whole processing chain.

The analysis is a bit involved, so we will not go into the details here, but will just focus on the results.

We can show that the output signal is of the form

\[ v(t) = k_f m(t) + n_d(t) \]

where \( n_d(t) \) is a (random) noise, independent of the message signal.

Output signal power therefore is \( k_f P \).

Output noise power: The PSD of the output noise is

\[ S_{N_d}(f) = \begin{cases} \frac{No_f^2}{A^2} & f \leq \frac{B_t}{2} \\ 0 & \text{otherwise} \end{cases} \]

Note that the noise at the output is no longer white! This is because we put white noise through a differentiator.

We are only concerned about noise within the signal BW of \( 2W \), which is usually smaller than the FM transmission BW of \( B_t \).

Thus average noise power at the output is

\[ \int_{-W}^{W} \frac{No_f^2}{A^2} df = \frac{2NoW^3}{3A^2} \]

And the output SNR is

\[ (SNR)_o = \frac{k_f^2 P}{2NoW^3} = \frac{3A^2 k_f^2 P}{2NoW^3} \]
The figure of merit for FM is:

\[
\frac{(SNR)_o}{(SNR)_c} \text{FM} = \frac{3k^2P}{W^2}.
\]

Recall Carson's rule for FM bandwidth:

\[
B_t \approx 2Af(1+\frac{1}{R}) = 2k_f A_m(1+\frac{1}{P})
\]

\[
Af = k_f A_m \quad \text{(Amplitude of modulating signal)}
\]

Thus \(B_t \approx k_f\) and increasing the transmission BW of a FM signal help increase the SNR output quadratically.

(Note \(B_t \neq W = B_t\) is the transmission BW
\(W\) is the message BW.
\(W\) is fixed, but you have control over \(B_t\).

Hence in an FM system, one can trade off BW for noise performance. Not so in AM.

- **FM capture and threshold effects:**

  1) Capture effect: noise in an FM system can also include interference.

  When interference is stronger than the wanted signal, the FM receiver will lock onto the stronger one, which is the interference.

  When interference and wanted signals are nearly equal in strength, the receiver fluctuates back and forth between them. This phenomenon is called the "capture effect."

  2) Threshold effect: because of being non-linear, FM receivers will breakdown if the carrier-to-noise ratio (CNR) is small. As the CNR decreases, we start to hear clicks, then crackling and spattering sounds, then the RX fails completely.
beyond a certain threshold on the CNR.

- FM threshold can be reduced by using some negative feedback (FMFB), or by using a phase-locked loop demodulator.

- FM pre-emphasis and de-emphasis:

  Noise power on FM demodulated signal is highest at high prep, where the signal power is usually lowest.

  Thus we can reduce the impact of noise by boosting the high frequency signal components during the modulation process, before noise is added.

  Then at the demodulator, we apply de-emphasis which has the opposite effect and restores the signal to its original form. This de-emphasis will dampen the noise at high frequency and help reduce the overall noise power at the demodulator output.

The pre-emphasis and de-emphasis can be non-linear. They are used in Dolby systems (Dolby-A, Dolby-B DBX systems).
Spectral response and spectral density of AM/DSB

1) Deterministic signals:

\[ s(t) = m(t) A_c \cos(2\pi f_c t) \]

\[ S(f) = \frac{A_c}{2} \left[ M(f-f_c) + M(f+f_c) \right] \]

2) Random signals:

\[ s(t) = m(t) A_c \cos(2\pi f_c t + \theta) \]

where \( m(t) \) is a random process, PSD \( S_M(f) \)
\( \theta \) is uniformly distributed on \([-\pi, \pi]\) and independent of \( m(t) \).

\[ R_S(f) = R_M(f) \frac{A_c^2}{2} \cos(2\pi f_c t) \]

\[ S_S(f) = \frac{A_c^2}{4} \left[ S_M(f-f_c) + S_M(f+f_c) \right] \]

\[ P_M = \int_{-W}^{W} S_M(f) \, df = \text{input power} \]

\[ P_S = \frac{A_c^2}{4} \int_{-\infty}^{\infty} \left[ S_M(f-f_c) + S_M(f+f_c) \right] \, df \]

\[ = \frac{A_c^2}{4} (P_M + P_M) = \frac{A_c^2}{2} \cdot P_m \quad \text{output power} \]
After coherent demodulation:

1) Deterministic signals:

\[ V(t) = S(t) \cos(2\pi fc t) \]
\[ = \frac{A_c}{2} m(t) + \frac{A_c}{2} m(t) \cos(2\pi fc t) \]

2) Stochastic/random signals:

\[ V(t) = S(t) \cos(2\pi fc t + \theta) \]
\[ = \frac{A_c}{2} m(t) + \frac{A_c}{2} m(t) \cos(2\pi fc t + 2\theta) \]

\[ R_v(t) = \frac{A_c^2}{4} R_m(t) + \frac{A_c^2}{4} R_m(t) \cdot \frac{1}{2} \cos(2\pi 2fc t) \]

\[ S_v(t) = \frac{A_c^2}{4} S_m(t) + \frac{A_c^2}{16} \left[ S_m(t - 2fc) + S_m(t + 2fc) \right] \]