Lecture 3:

The inverse relationship between time and frequency.

- From the FT properties, we see that if the signal changes in one domain (say time), its description in the other domain (frequency) will change in an inverse manner.

- If a signal is time limited, its frequency content will expand all frequencies.
  Vice versa, if a signal is band limited, it must be spanning infinite time.

A signal cannot be strictly limited in both freq & time.

- Bandwidth: contains "significant" spectral content of the signal for positive frequencies.

This definition is mathematically imprecise. Thus there are multiple ways to define bandwidth. We outline 3 here.

1) Mill-to-null bandwidth: BW associated with the mainlobe of a frequency response.
   - For low-pass signal: BW = \frac{1}{2} of total mainlobe width (since only account for positive freq.)
   - For band-pass signal: BW = width of mainlobe.

\[ G(f) \]

\[ f \]

\[ \text{mill-to-null BW} \]
\[ \text{low-pass signal} \]

\[ \text{mill-to-null BW} \]
\[ \text{band-pass signal} \]
1) 3dB bandwidth: measured by the point that amplitude spectrum drops to $\sqrt{2}$ of its peak value (i.e. power drops by 3dB).

\[ 1 |G(f)| \]
\[ \sqrt{2} \]
\[ f \text{ dB bandwidth,} \]
\[ \text{low-pass signal} \]

2) RMS bandwidth: defined as the square root of the second moment of squared amplitude spectra.

\[ W_{\text{rms}} = \left( \frac{\int \sqrt{G(f)}^2 df}{\int |G(f)|^2 df} \right)^{1/2} \rightarrow \text{for low-pass signal} \]

For band-pass signal the center the spectrum around the carrier frequency instead of around zero.

\[ W_{\text{rms}} \text{ is mathematically tractable but is not easily measured in the laboratory.} \]

- Time-bandwidth product.

For any family of pulse signals that are time-scale of each or other, the time-bandwidth product is a constant.

\[ (\text{duration}) \cdot (\text{bandwidth}) = \text{constant} \]

This relation holds for any definition of bandwidth; different definition merely changes the value of the constant.

Remember the time scaling property:

- compression in time = expansion in freq.
- expansion in time = compression in freq.
Transmission of signals through linear systems.

A system: a physical device that produces an output signal in response to an input signal.

Linear systems: response to a sum of input signals is equal to the sum of individual responses (principle of superposition).

Examples of linear systems: filters, communication channels.

To evaluate the effect of a linear system on the input signals, we can use either the time response or the frequency response.

- Time response: response of the system to a \( \delta(t) \) input (assuming zero initial condition).
- Frequency response: \( y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \)
A linear system is completely specified by its time response $h(t)$. Different system properties can be derived from $h(t)$:

- **Causality**: A system is causal if and only if $h(t) = 0$ for $t < 0$.

- **Stability**: bounded input bounded output (BIBO). A LTI system is BIBO if and only if $\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$, i.e., $h(t)$ is absolutely integrable.

- **Frequency response**:
  
  $Y(f) = H(f) \cdot X(f)$  \hspace{3cm} \text{convolution property}

  $H(f)$: transfer function of the system

  $H(f) = |H(f)| \cdot e^{j\phi(f)}$  \hspace{2cm} \text{phase response}

  \hspace{1cm} \text{amplitude response}

  The definitions of bandwidth also apply to a system as well.

- A system can be low-pass (if $|H(f)|$ concentrates around zero freq.) or band-pass (if $|H(f)|$ concentrates around a carrier freq.).

- **Filters**:
  
  - A filter is a frequency selective device used to limit the spectrum of a signal to some specified band of frequencies.

  - A filter can be low-pass, high-pass, band-pass or band-stop, depending on the frequency range it allows to pass through. In common systems, we mostly concern with lowpass and bandpass.
A communication channel can be viewed as a filter. When a signal is transmitted through a channel (filter), some delay is introduced. The delay is dictated by the phase response. If the phase response of the channel is linear in frequency, then the output signal has the same delay at all frequencies.

But often the phase response $\beta(f)$ is non-linear. Then define the group delay as

$$\Delta(t) = \frac{d}{df} \beta(t).$$

This group delay gives the delay at the output as a function of frequency.
Design of filters - the basics

An ideal filter with a sharp cut-off in frequency and a zero stop-band is non-causal and not realizable in practice.

\[ |H(f)| \]

\[ \text{pass-band} \]

\[ \text{stop-band} \]

\[ \text{transition band} \]

\[ \delta_1 \]

\[ \delta_2 \]

ideal low-pass filter

practical low-pass filter

The Paley-Wiener criterion implies that any causal, realizable filter cannot have a frequency response that is zero for a band of frequencies. (The frequency response can be zero at singular points of frequency; that's OK).

For that matter of fact, any realizable causal filter cannot have a frequency response that is completely flat over a band of frequencies.

Therefore, all practical, realizable filter (analog or digital) will have alternations in both pass-band and stop-bands.

And the frequency response can extend to infinity.

It is just a matter of how low you can make the stop-band in comparison to the pass-band amplitude.

There is also a trade-off between how sharp the transition band and how low the stop-band are.

For the same type of filters, the narrower the transition band, the higher the side-lobes (stop-band) will be.

There are many types of filters, but we will not go into that.