Topic 4  Noise in CW modulation systems.

- We want to analyze and understand performance of different CW modulation techniques studied so far when there is noise.

- In order to perform such an analysis, we need a receiver model.

\[ s(t) + \text{Noise } w(t) \rightarrow \sum \rightarrow \text{Bandpass filter} \rightarrow x(t) \rightarrow \text{Demodulator} \rightarrow \text{output signal } \hat{r}(t) \]

- Front-end receiver noise model:

  - Noise \( w(t) \) is additive, white and Gaussian,
  - Usually called additive white Gaussian noise or AWGN.

  What do those characteristics really mean?

+ Some probability and random process background:

  - \( w(t) \) is a random process:
    - At each time instant, it is a random variable
    - The realization of this random variable changes with time

Since \( w(t) \) is a random function of time, its statistical averages such as mean, correlation, and covariance could be functions of time.

- Its average power could also vary with time.

Then, how do we characterize its "frequency response"?
Power spectral density - average power per unit bandwidth.

- For a special class of random processes, called "wide-sense stationary" processes, then the power spectral density does not change over time.
- We can use it to perform frequency domain analysis of random signals.

Specifically, let \( X(t) \) be a random process.

- Its realization is called a sample function, \( x(t) \).
- At each time \( t \), \( X(t) \) is a random variable.

\[
\mu_X(t) = E[X(t)] \quad \text{(first moment)}
\]

auto-correlation function

\[
R_X(t_1, t_2) = E[X(t_1)X(t_2)] \quad \text{(second moment)}
\]

Wide-sense stationary processes:

\[
\mu_X(t) = \mu_X \quad \forall t.
\]

\[
R_X(t_1, t_2) = R_X(t_1 - t_2) \quad \forall t_1, t_2.
\]

[or write \( R_X(t, t+c) = R_X(c) \quad \forall c \).]

Some properties of the auto-correlation function:

i) \( R_X(0) = E[X^2(t)] \)

ii) \( R_X(\tau) = R_X(-\tau) \) an even function

iii) \( |R_X(\tau)| \leq R_X(0) \) maximum magnitude at \( \tau = 0 \).

\( R_X(\tau) \) indicates how fast the process varies with time.

It shows the interdependency between 2 points at \( \tau \) sec apart.
Power spectral density:
Recall for a deterministic function \( x(t) \), we can use its Fourier transform to analyze the frequency response \( X(f) \).

For a random process \( X(t) \), each realization \( x(t) \) has its own Fourier transform. Then how do we characterize the whole ensemble of realizations/sample functions?

- Deterministic: \( x(t) \xrightarrow{FT} X(f) \)
- Random process: \( X(t) \xrightarrow{statistical \ average} X(f) \)

Autocorrelation:
\[
R_x(t) \xrightarrow{FT} S_x(f) \quad \text{Power spectral density}
\]

\[
S_x(f) = \int_{-\infty}^{\infty} R_x(t) e^{-j2\pi ft} dt 
\]

\[
R_x(t) = \int_{-\infty}^{\infty} S_x(f) e^{j2\pi ft} df
\]

Some properties of \( S_x(f) \):

i) \( \mathbb{E}[x^2(t)] = \int_{-\infty}^{\infty} S_x(f) df \) = average signal power

average power in random (stochastic) \( X(t) \)

Thus \( S_x(f) \) is the power "density" \( \kappa \) power per unit bandwidth.

ii) \( S_x(f) \geq 0 \quad \forall \, f \) this makes sense for power density.
The reason is that $S_x(f) \approx E[P(f)^2]$
where $P(f)$ is the Fourier transform of $X(t)$

(iii) $S_x(f) = S_x(-f)$, if $X(t)$ is real-valued.

Example: Sineoidal signal with random phase

$X(t) = A \cos(2\pi f_c t + \Theta)$

$\Theta$: uniform in $[-\pi, \pi]$

$P_\Theta(\theta) = \frac{1}{2\pi}, -\pi \leq \theta \leq \pi$

$A, f_c$: constants.

Can show $R_x(t) = \frac{A^2}{2} \cos(2\pi f_c t)$

and $S_x(f) = \frac{A^2}{4} \left[ \delta(f - f_c) + \delta(f + f_c) \right]$

Total "area" under $S_x(f)$

is $\frac{A^2}{2}$ which is the power of the sinusoid.

Transmission of a random process through a linear system.

random input signal $X(t) \rightarrow [h(t), H(f)] \rightarrow Y(t)$

Assume $X(t)$ is wide-sense stationary

$Y(t) = X(t) \ast h(t)$

We can show that

$S_Y(f) = |H(f)|^2 \cdot S_x(f)$

This is why the power-spectral density is important and easy to use in analysis.
Additive white Gaussian noise (AWGN)

- Let's get back to our model for the receiver noise.

- Luckily, w(t) can be modeled as wide-sense stationary.
  - White noise: The PSD $S_w(f)$ is constant for all freq.
    $$S_w(f) = \frac{N_0}{2} + f$$
  
  Note that this power density is defined for both positive and negative frequencies.
  Thus the power per unit bandwidth measured at the front-end of the receiver is $N_0$.

- Autocorrelation:
  $$R_w(t) = \frac{N_0}{2} \delta(t)$$

  - White noise has any two different samples uncorrelated, no matter how close in time.

- Gaussian noise: $X(t)$ is a Gaussian random process if any set of observations $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly Gaussian, for all $n$ and $t_1, t_2, \ldots, t_n$.

  If we apply a Gaussian process $X(t)$ to a stable linear system, then the output $Y(t)$ is also a Gaussian process.

  The Gaussian process can model many physical phenomena with good accuracy, and it is easy to analyze.
Lecture 10:

- White Gaussian noise:
  Any two samples in time are independent, no matter how close in time.

- We can accurately model receiver noise as AWGN.

\[ s(t) \rightarrow \text{BPF} \rightarrow x(t) \rightarrow \text{Demod} \rightarrow w(t) \rightarrow y(t) \]

Since we use a BPF to extract the portion of frequencies that contain our signal, we really only need to model the noise within this passband.

Average noise power
\[ = N_0 B_t \]

The filtered noise can be modeled as a narrowband noise (since transmission bandwidth \( B_t \ll f_c \)).

\[ n(t) = n_1(t) \cos(2\pi f_c t) - n_2(t) \sin(2\pi f_c t) \]

\( n_1(t) \) - in-phase noise component
\( n_2(t) \) - quadrature noise component

Define input (or channel) signal-to-noise ratio (SNR)

\[ (\text{SNR})_c = \frac{\text{average power of modulated signal}}{\text{average noise power in the message bandwidth}} \]

(as measured at the input of the receiver).

Output SNR:

\[ (\text{SNR})_o = \frac{\text{average power of demodulated signal}}{\text{average noise power}} \]

(as measured at the receiver output).

Assuming same (SNR)$_c$, we are going to analyze and compare (SNR)$_o$. 
Noise in DSB-SC receivers

This is the simplest case for analysis, so we will start here.

We assume coherent detection with perfect synchronization.

\[
\begin{align*}
S(t) & \rightarrow \Sigma \rightarrow \text{BPF} \rightarrow x(t) \rightarrow \times \rightarrow v(t) \rightarrow \text{LPF} \rightarrow y(t)
\end{align*}
\]

- Coherent detection
- \(x(t) = \cos(2\pi f_c t) m(t)\)
- \(v(t) = \cos(2\pi f_c t) n(t)\)

**Channel SNR (at input of the receiver)**

\[
S(t) = A_c \cos(2\pi f_c t) m(t)
\]

- Assume that \(m(t)\) is band-limited to a bandwidth \(W\).
- Let \(P\) be the average power of \(m(t)\).

- The noise power within the bandwidth of the message signal is \(W N_0\) (assuming noise PSD of \(N_0/2\)).

- The average power of the modulated signal is \(\frac{A_c^2}{2} P\).

**How do we compute the average power of a modulated signal?**

Since our communication lies to work for many different message signals, for each message signal \(m(t)\), we can think of it as a realization (a sample function) of a random process.

The ensemble of all realizations of this process is the set of all possible input message signals.

Often we will assume this random process is stationary (and hence is wide-sense stationary) and has zero mean (so that it does not contain any DC power).
Suppose we denote the input random process as $X(t)$, then the modulated signal as received at the frontend receiver is also a random process

$$Y(t) = X(t) \cos(2\pi fc t + \Theta)$$

Here $\Theta$ is uniform $[-\pi, \pi]$ to model the random phase variation by going through the channel. Note that the input signal $X(t)$ and the carrier are independent.

Then we can compute the auto-correlation:

$$R_Y(f) = E[Y(t)Y(t+\tau)]$$

$$= E[X(t)X(t+\tau)\cos(2\pi fc t + \Theta)\cos(2\pi fc (t+\tau) + \Theta)]$$

$$= R_X(\tau) E[\cos(2\pi fc t + \Theta)\cos(2\pi fc (t+\tau) + \Theta)]$$

$$= \frac{1}{2} R_X(\tau) E[\cos(2\pi fc \tau) + \cos(2\pi fc (2t+\tau) + 2\Theta)]$$

$$= \frac{1}{2} R_X(\tau) \cos(2\pi fc \tau)$$

The power spectral density is

$$S_Y(f) = \frac{1}{4} \left[ S_X(f-fc) + S_X(f+fc) \right]$$

Thus the average power of the modulated signal is

$$P_Y = \frac{1}{2} P_X$$

(The $\frac{1}{2}$ factor comes from the cosine carrier.)

1) Back to our DSB-SC signal:

$$\text{SNR}_c = \frac{\frac{1}{2} A_0^2 P}{W N_0}$$

1) Next we need to compute the SNR of the demodulated signal $y(t)$. 
Following the processing chain in the receiver:

\[ x(t) = s(t) + n(t) \]
\[ = A_c \cos(2\pi f_c t) m(t) + n_0(t) \cos(2\pi f_c t) - n_0(t) \sin(2\pi f_c t) \]

\[ v(t) = x(t) \cos(2\pi f_c t) \]
\[ = \frac{1}{2} A_c m(t) + \frac{1}{2} n_0(t) \]
\[ + \frac{1}{2} (A_c m(t) + n_0(t)) \cos(2\pi f_c t) - \frac{1}{2} A_c n_0(t) \sin(2\pi f_c t) \]

After LPF, then we obtain

\[ y(t) = \frac{1}{2} A_c m(t) + \frac{1}{2} n_0(t) \]

Only the in-phase noise component left!

The signal power in \( y(t) \) is \( \frac{A_c^2}{4} P \).

The noise power in \( y(t) \) is \( \left(\frac{1}{2}\right)^2 \cdot 2N_0W = \frac{N_0}{2} \)

because the bandwidth of the BPF is \( 2W \).

Thus, the output SNR is

\[ (SNR)_o = \frac{\frac{1}{4} A_c^2 P}{\frac{1}{2} W N_0} = \frac{A_c^2 P}{2N_0 W} \]

Then the figure of merit for DSB-SC systems is:

\[ (SNR)_o \left| \frac{(SNR)_c}{DSB-SC} \right| = 1 \]

\[ \text{Noise in AM receiver:} \]

Consider the full AM signal that can use envelope detector in the receiver:

\[ s(t) = A_c \left[ 1 + k \cdot m(t) \right] \cos(2\pi f_c t) \]
Following the processing chain in the receiver:

\[ r(t) = s(t) + n(t) \]

\[ = A_c \cos(\omega_c t) m(t) + n_1(t) \cos(\omega_c t) - n_2(t) \sin(\omega_c t) \]

\[ v(t) = r(t) \cos(\omega_c t) \]

\[ = \frac{1}{2} A_c m(t) + \frac{1}{2} n_1(t) \]

\[ + \frac{1}{2} \left( A_c m(t) + n_1(t) \right) \cos(2\pi f_c t) - \frac{1}{2} A_c n(t) \sin(2\pi f_c t) \]

After LPF, then we obtain

\[ y(t) = \frac{1}{2} A_c m(t) + \frac{1}{2} n_1(t) \]

Only the in-phase noise component left!

The signal power in \( y(t) \) is \( \frac{A_c^2}{4} P \).

The noise power in \( y(t) \) is \( \left( \frac{1}{2} \right)^2 2N_0 = \frac{N_0}{2} \), because the bandwidth of the BPF is \( 2W \).

Thus the output SNR is

\[ (SNR)_o = \frac{\frac{1}{2} A_c^2 P}{\frac{1}{2} W N_0} = \frac{A_c^2 P}{2N_0} \]

Then the figure-of-merit for DSB-SC systems is:

\[ \frac{(SNR)_o}{(SNR)_C} \bigg|_{\text{DSB-SC}} = 1 \]

\(-\)

- Noise in AM receiver:

Consider the full AM signal that can use an envelope detector in the receiver:

\[ s(t) = A_c \left[ 1 + k a m(t) \right] \cos(\omega_c t) \]
AM signal  \( s(t) \)  
\[ \sum \]  
\[ \text{BPF} \]  
\[ x(t) \]  
\[ \text{Envelope detector} \]  
\[ y(t) \]  
\[ \text{output signal} \]

**Channel SNR:**
- Average power of the carrier component = \( \frac{A_c^2}{2} \).
- Average power of the information-bearing component = \( \frac{\frac{1}{2} A_c^2 P}{\text{NoW}} \).
- Average power of noise within the message bandwidth (NoW).

Thus \( \text{(SNR)}_x = \frac{1}{2} \frac{A_c^2 + \frac{1}{2} A_c^2 P}{\text{NoW}} \).

**Output SNR:** Again follow the Rx processing chain:
- Filtered signal
  \[ x(t) = s(t) + n(t) \]  
  \[ = \left[ A_c + A_c k_m(t) + n_I(t) \right] \cos(2 \pi f_c t) - n_d(t) \sin(2 \pi f_c t) \]  
- Output signal \( y(t) \) is the envelope of \( x(t) \).

Since the in-phase and quadrature components are orthogonal, the envelope of \( x(t) \) is

\[ y(t) = \left( (A_c + A_c k_m(t) + n_I(t))^2 + n_d(t)^2 \right)^{\frac{1}{2}} \]

Assume that the carrier power is much higher than the noise power on average, which is often the case, we can,
Simplify the expression for $y(t)$ as:

$$y(t) \approx A_c + A_{ck} m(t) + \eta(t)$$

The DC component $A_c$ can be replaced by a blocking capacitor, effectively no reflection to $m(t)$.

Thus the output signal power is $A_{ck}^2 P$ and the output noise power is approximately $2NF_b W$ (because the output after bandpass has bandwidth $2W$):

$$\Rightarrow (SNR)_o \approx \frac{A_{ck}^2 P}{2NF_b}$$

Note that $(SNR)_o$ above is approximate under 2 conditions:
- average noise power is small compared to carrier-power
- amplitude sensitivity $k_c$ is such that percentage modulation $\% \leq 100\%$

Then the figure of merit for AM systems is:

$$\frac{(SNR)_o}{(SNR)_{c \text{ AM}}} = \frac{k_c^2 P}{1 + k_c^2 P} < 1 \quad (\text{AM wastes power in transmitting the carrier})$$

Threshold Effect:
What if the noise power increases, or the carrier power decreases, such that the above approximation no longer holds?

Let's look at the case when the noise dominates the carrier. It is better in this case to express the noise in its polar form:

$$n(t) = r(t) \cos(2\pi f_c t + \theta(t))$$

$\theta(t)$ is the phase, which is uniformly random between $(-\pi, \pi)$. 
Based on the above phasor diagram, the output is approx:
\[ y(t) = r(t) + A_c[1+k_m u(t)] \cos \phi(t) \]
\[ = r(t) + A_c \cos \phi(t) + Ak_m u(t) \cos \phi(t) \]

But we cannot really extract the message signal \( u(t) \) back from this \( y(t) \) output since the phase \( \phi(t) \) is unknown and random.

Thus at very low carrier-to-noise ratio (<<1), there is a complete loss of information in AM system.

This effect is called the "threshold effect." That is, below a certain threshold on the carrier-to-noise ratio, the receiver performance deteriorates rapidly and information is lost.

This threshold effect is common in non-linear receivers such as the envelope detector. It does not apply to the coherent detector.

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**Noise in FM receivers:**

- FM signals contain information in the phase.
- The amplitude of FM signals is constant. Thus any amplitude variation at the receiver input is due to noise (or interference).
- Thus in the FM receiver, the input signal is clipped and filtered before demodulation.
Noise after BPF:
\[ n(t) = n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) = r(t) \cos\left[2\pi f_c t + \varphi(t)\right] \]
where
noise envelope \[ r(t) = \left( n_I^2(t) + n_Q^2(t) \right)^{1/2} \rightarrow \text{Rayleigh, distributed} \]
noise phase \[ \varphi(t) = \tan^{-1}\left( \frac{n_Q(t)}{n_I(t)} \right) \rightarrow \text{uniform [-\pi, \pi]} \]

FM signal:
\[ s(t) = A_c \cos\left[2\pi f_c t + 2\pi f_m \int_0^t m(t) \, dt\right] \]
\[ = A_c \cos\left[2\pi f_c t + \varphi(t)\right]. \]

- Channel SNR:
  This is straightforward to compute.
  - FM signal power = \( \frac{1}{2} A_c^2 \).
  - Noise power in the message BW is \( N_0 \)\( W \)\.
  \[ \text{(SNR)}_c = \frac{A_c^2}{2N_0 W} \]

- Output SNR:
  This part is much trickier to compute and requires some assumptions.
  In particular, we will again assume that the carrier power is much larger than the noise power
  i.e., carrier-to-noise ratio > 71.
We are interested in the noise power at the output, after going through the whole processing chain.

The analysis is a bit involved so we will not go into the details here, but will just focus on the results.

We can show that the output signal is of the form
\[ v(t) = k_f m(t) + n_d(t) \]
where \( n_d(t) \) is a (random) noise, independent of the message signal.

Output signal power therefore is \( k_f^2 P \).

Output noise power: The PSD of the output noise is
\[ S_{n_d}(f) = \begin{cases} \frac{N_0 f^2}{A_c} & f \leq \frac{B_t}{2} \\ 0 & \text{otherwise} \end{cases} \]

Note that the noise at the output is no longer isolite! This is because we put isolite noise through a differentiator.

We are only concerned about noise within the signal BW of \( 2W \), which is usually smaller than the FM transmission BW of \( B_t \).

Thus, average noise power at the output is
\[ \int_{-W}^{W} \frac{N_0 f^2}{A_c} df = \frac{2 N_0 W^3}{3 A_c^2} \]

And the output SNR is
\[ (SNR)_o = \frac{k_f^2 P}{2 N_0 W^3} = \frac{3 A_c^2 k_f^2 P}{2 N_0 W^3} \]
The figure of merit for FM is:

\[
\frac{(SNR)_0}{(SNR)_c} = \frac{3kT P}{W^2}
\]

Recall Carson's rule for FM bandwidth:

\[
B_T \approx 2Af \left(1 + \frac{1}{f_c^2} \right) = 2kT A_m \left(1 + \frac{1}{f_c^2} \right)
\]

\[
Af = kT A_m 
\]

(\text{Am: amplitude of } m(t))

Thus, \(B_T \propto kT\), and increasing the transmission BW of an FM signal helps increase the SNR output quadratically!.

(Note: \(B_T \neq W\) or \(B_T\) is the transmission BW

\(W\) is the message BW.

\(W\) is fixed, but you have control over \(B_T\).

Hence in an FM system, one can trade off BW for noise performance. Not so in AM.

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**FM capture and threshold effects:**

1) Capture effect: noise in an FM system can also include interference.

   When interference is stronger than the wanted signal, the FM receiver will lock onto the stronger one, which is the interference.

2) Threshold effect: because of being non-linear, FM receivers will breakdown if the carrier-to-noise ratio (CNR) is small. As the CNR decreases, we start to hear clicks, then crackling and spurring sounds, then the RX fails completely.
Beyond a certain threshold on the CNR.

FM threshold can be reduced by using some negative feedback (FMFB), or by using a phase-locked loop demodulator.

FM pre-emphasis and de-emphasis:

Noise power in FM demodulated signal is highest at high freq. where the signal power is usually lowest.

Thus we can reduce the impact of noise by boosting the high frequency signal components during the modulation process, before noise is added.

Then at the demodulator, we apply de-emphasis which has the opposite effect and restores the signal to its original form. This de-emphasis will dampen the noise at high frequency and help reduce the overall noise power at the demodulator output.

The pre-emphasis and de-emphasis can be non-linear. They are used in Dolby systems. (Dolby-A, Dolby-B, DBX systems).
Spectral response and spectral density of AM/DSB

1) Deterministic signals:

\[ s(t) = m(t) A_c \cos(2\pi f_c t) \]

\[ S(f) = \frac{A_c}{2} \left[ M(f-f_c) + M(f+f_c) \right] \]

2) Random signals:

\[ s(t) = m(t) A_c \cos(2\pi f_c t + \Theta) \]

where \( m(t) \) is a random process, PSD \( S_M(f) \), \( \Theta \) is uniform \([-\pi, \pi]\), independent of \( m(t) \).

\[ R_S(f) = R_M(f) \frac{A_c^2}{2} \cos(2\pi f_c t) \]

\[ S_S(f) = \frac{A_c^2}{4} \left[ S_M(f-f_c) + S_M(f+f_c) \right] \]

\[ P_M = \int_{-W}^{W} S_M(f) \, df = \text{input power} \]

\[ P_S = \frac{A_c^2}{4} \int_{-\infty}^{\infty} \left[ S_M(f-f_c) + S_M(f+f_c) \right] \, df \]

\[ = \frac{A_c^2}{4} \left( 2P_M + P_M \right) = \frac{A_c^2}{2} \cdot P_M \quad \text{output power} \]
After coherent demodulation:

1) Deterministic signals:

\[ u(t) = S(t) \cos(2\pi f_c t) \]
\[ = \frac{A_c}{2} m(t) + \frac{A_c}{2} m(t) \cos(2\pi f_c t) \]

2) Stochastic/random signals:

\[ u(t) = S(t) \cos(2\pi f_c t + \theta) \]
\[ = \frac{A_c}{2} m(t) + \frac{A_c}{2} m(t) \cos(2\pi f_c t + \theta) \]

\[ R_v(t) = \frac{A_c^2}{4} R_m(t) + \frac{A_c^2}{4} R_m(t) \frac{1}{2} \cos(2\pi f_c t) \]

\[ S_v(t) = \frac{A_c^2}{4} S_m(t) + \frac{A_c^2}{16} [S_m(t-2f_c) + S_m(t+2f_c)] \]