Diversity Techniques

1. We see that fading makes the received SNR fluctuate. This fluctuation in the SNR not only reduces the capacity (compared to an AWGN channel with the same average SNR), it also makes signal detection/decoding error prone.

2. Diversity techniques combine independent fading signal paths in order to improve the error performance. The main idea is that independent signal paths have a low probability of experiencing deep fades simultaneously.

   Hence correctly combining these paths leads to a signal SNR with less variation, hence decoding has lower error probability.

3. Diversity can be applied over the time, frequency or spatial dimensions.

   - Time diversity: transmit the same signal at different times separated by more than the channel coherence time.

   - Frequency diversity: transmit the same narrowband signal over different carrier frequencies separated by more than the channel coherence bandwidth.

   - Spatial diversity: use an antenna array to transmit or receive signals. The separation among the antennas need to be at least half a wavelength (carrier wavelength) to obtain uncorrelated or independent fading at different antennas.
Polarization diversity: transmit signals on two orthogonal polarizations. Only two diversity branches are available.

Diversity performance includes a power gain (array gain) and a diversity gain (change in the slope of the error probability vs. SNR).

Diversity performance depends on the combining technique.

\[ \text{Diversity performance} \]

\[ \log \text{scale} \]

\[ 10^{-1} \]

\[ 10^{-2} \]

\[ 10^{-3} \]

\[ 10 \]

\[ 5 \]

\[ 10 \]

\[ 15 \]

\[ 20 \text{SNR (dB)} \]

\[ \text{Horizontal shift indicates a power gain} \]

\[ \text{slope at high SNR} = \text{diversity order} \]

1) Receiver diversity combining models:

2) System model:
   - Take multiple antennas as the example, but the model applies equally well to frequency and time diversity.
   - The receiver receives M independent branches of the same transmit signal.
   - The question is how to combine these M independent
received signals, and what is the performance of such a combining method. The received signal can be written as a vector

\[ r = h s + n \]

where

\[ h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{j\theta_1} \\ \alpha_2 e^{j\theta_2} \\ \vdots \\ \alpha_M e^{j\theta_M} \end{bmatrix} \quad ; \quad r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_M \end{bmatrix} \quad ; \quad n = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_M \end{bmatrix} \]

and \( s \) is a complex scalar representing the transmit symbol.

The combining technique is usually linear: each branch is scaled by a complex weighting factor \( w_i \) and then sum. The combining weight vector can be written as

\[ w = [w_1 \ w_2 \ \cdots \ w_M]^T \]
The combiner output is a single value

\[ y = \mathbf{w}^T \mathbf{r} = \sum_{i=1}^{M} w_i r_i = \mathbf{w}^T \mathbf{h.s} + \mathbf{w}^T \mathbf{n} \]

Different ways of designing the weight vector lead to different combining techniques.

To measure performance, we examine the received SNR after combining:

Suppose \( s \) has average power \( P \) and noises \( n_i \overset{\in}{\sim} \text{CN}(0, \sigma^2) \) and are independent at different branches.

\[ r_i = \frac{P}{\sigma^2} |h_i|^2 \] is the received SNR at branch \( i \).

The SNR after combining is

\[ \gamma = \frac{P}{\sigma^2 (\sum |w_i|^2)} \cdot \left| \sum w_i h_i \right|^2 \]

Thus combined SNR is still a random quantity. We look at several attributes:

i) The average SNR:

\[ \overline{\gamma} = \frac{P}{\sigma^2 (\sum |w_i|^2)} E \left[ \left| \sum w_i h_i \right|^2 \right] \]

This \( \overline{\gamma} \) gives the average SNR gain or array gain.

ii) The error performance at high SNR:
The combing weight vector contains only one entry.

This method only selects the output with the highest
instantaneous SNR.

The error probability falls off with increasing (average) SNR.

Usually, this error probability can be evaluated to be

\[ P_e = Pr(Y < \tilde{y}) = \int_{-\infty}^{\tilde{y}} f_\tilde{y}(x) \, dx \]

where \( c \) is a constant, the diversity order.

The diversity order for a given target SNR is
an indication of the symbol error probability and is equal
to \( \tilde{y} \), denoted as \( \tilde{y} \).
\[ Y^* = \max_i \{ \gamma_i \} = \max_i \left\{ \frac{P}{\sigma^2} |h_i|^2 \right\} . \]

Suppose error occurs when the received SNR is below \( \gamma_0 \), which is the same as the cumulative distribution function (CDF).

For selection combining, all \( M \) branches must have received SNR below \( \gamma_0 \) for an error to occur.

The CDF of \( Y^* \) is
\[
F_{Y^*}(y) = R_c(Y^* < y) = R_c \left( \max_i \gamma_i < y \right)
\]
\[
= \prod_i R_c(\gamma_i < y) \quad \text{because of uncorrelated branches.}
\]
\[
= \left[ R_c(\gamma_i < y) \right]^M \quad \text{if all } \gamma_i \text{ are identically distributed.}
\]

Example: For Rayleigh fading,
\[
f_{\gamma}(\gamma) = \frac{1}{\gamma} e^{-\gamma/\delta}
\]

Then for a single branch (no combining)
\[
P_o(\gamma_0) = 1 - e^{-\gamma_0/\delta}
\]

With selection combining, the outage probability becomes:
\[
P_o(\gamma_0) = \prod_{i=1}^M \left( 1 - e^{-\gamma_0/\delta} \right)
\]
\[
= \left( 1 - e^{-\gamma_0/\delta} \right)^M \quad \text{if all } \gamma_i = \bar{\gamma}.\]
+) An even simpler scheme is threshold combining: find the first branch that has SNR above a given threshold $j$, then switch to another branch when SNR of the current branch falls below $j$.

This method is also called switch and stay combining. It has the same outage probability as selection combining provided the threshold is optimised.

+) Maximal Ratio Combining: (MRC).

The goal of MRC is to maximise the output SNR $\gamma_B$:

$$\gamma_B = \frac{P}{\sigma^2} \frac{\left| \sum_i w_i h_i \right|^2}{\sum_i |w_i|^2}$$

By Cauchy-Schwarz inequality:

$$\left| \sum_i w_i h_i \right|^2 \leq (\sum_i |w_i|^2) (\sum_i |h_i|^2)$$

"=" iff $w_i = c \cdot h_i^*$ for some scaling factor $c$.

Thus the optimal combining weight vector can be chosen as $w = h^*$ (where $^*$ stands for complex conjugate).

Then the resulting output SNR is:

$$\gamma_B = \frac{P}{\sigma^2} \left( \sum_i |h_i|^2 \right) = \sum_i |h_i|^2$$

→ Output SNR increases linearly with the number of combining branches.
MRC combiner gives the best output SNR and hence the largest power (or array) gain.

The diversity order of MRC receiver is also M.

To compute the outage probability, we need to determine the distribution of the output SNR $J_Z$.

In general, this distribution can be derived from the characteristic function or moment generating function of the channel power gain (fading distribution).

If fading is Rayleigh, then $J_Z$ has a chi-squared distribution ($X^2$) with $2M$ degrees of freedom.

$$ f_{JZ}(j) = \frac{j^{M-1} e^{-j/\sigma}}{\sigma^M (M-1)!}, \quad j \geq 0 $$

and the outage probability can be computed in closed form:

$$ P_o = Pr(J_Z < \delta_0) = 1 - e^{-\delta_0/\sigma} \sum_{i=1}^{M} \frac{(\delta_0/\sigma)^i}{(i-1)!} $$

Equal gain combining:

As MRC requires the knowledge of the instantaneous channel gain on each branch, a simpler scheme is to only match the phase of the channel (co-phasers) then combines with equal weights.

Let $\hat{h_i} = \alpha_i e^{j\phi_i} \rightarrow W_i = e^{-j\phi_i}$.

The output SNR is

$$ J_Z = \frac{P}{\sigma^2} \left( \sum |h_i|^2 \right)^2.$$