MIMO Systems:

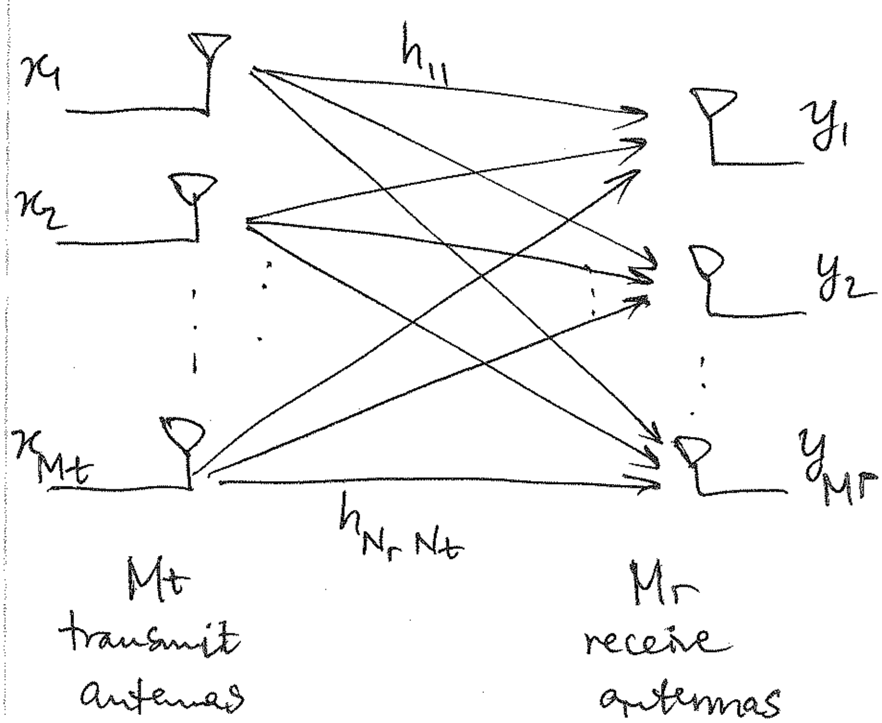
- MIMO communication was a breakthrough for wireless research during the last decade (1998-2008)

- It is now considered in all current and future wireless standards / system design.

- MIMO techniques are strong candidates for future systems such as 5G cellular.

- We will study multiple aspects of MIMO in this course, from theoretical capacity to practical transmission strategies and their performance.

V. Narrowband (flat fading) MIMO channel model.



Consider a MIMO system with M_t transmit antennas and M_r receive antennas.

The input_output signal model can be expressed in a vector form as

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{Mr} \end{bmatrix} = \begin{bmatrix} h_{11} & \cdots & h_{1Mt} \\ h_{21} & \cdots & h_{2Mt} \\ \vdots & & \\ h_{Mr1} & \cdots & h_{MrMt} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_{Mt} \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_{Mr} \end{bmatrix}$$

or as

$$y = Hx + n \quad \begin{cases} y \in \mathbb{C}^{Mr \times 1} : \text{vector of size } Mr \\ x \in \mathbb{C}^{Mt \times 1} : \text{vector " } Mt \\ H \in \mathbb{C}^{Mr \times Mt} : \text{channel matrix} \\ n \in \mathbb{C}^{Mr \times 1} : \text{noise vector} \end{cases}$$

- Here the MIMO channel is modeled as a complex valued matrix of size $Mr \times Mt$.

For fading channels, each entry is a random variable, often with identical distribution.

The entries of H are independent if there is enough antenna separation at the transmitter and receiver. Sometimes antennas correlation is assumed as a general case.

- The noise vector n is usually assumed to contain i.i.d (independent and identically distributed) components:

$$n \sim CN(0, I.\sigma_n^2).$$

- The transmit signal vector is subject to a sum power constraint across all antennas. Since the transmit signal is model as a random signal, this sum power constraint is expressed as

$$\mathrm{tr}\left(E[xx^*]\right) \leq P$$

, where x^* is the conjugate transposed vector.

Often we denote the input covariance matrix as

$$R_x = E[xx^*] = E\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{M_t} \end{bmatrix} \begin{bmatrix} x_1^* & x_2^* & \cdots & x_{M_t} \end{bmatrix}\right)$$

and the power constraint becomes

$$\mathrm{tr}(R_x) \leq P.$$

- Again the capacity of a MIMO wireless channel depends on whether the channel is known at the Rx, Tx or both.

- When the channel is known at both the Tx and Rx, the capacity is the same as that of a deterministic MIMO channel and is achieved by singular vector decomposition. This result is known since the 60's.

- When the channel is known only at the Rx, it is shown that, quite surprisingly, the capacity can still scale linearly with the number of antennas (specifically as $\min(M_t, M_r)$). This result was established around 1998/1999 and fueled an excitement about the application of MIMO in wireless. Since then it has come a long way.

2/. Capacity of a MIMO channel with CSIT & CSIR:

- For a given input signal design with covariance matrix R_x, the capacity of a MIMO channel with CSI at the receiver is given by:

$$C(R_x) = \log_2 \det\left[I + \frac{1}{\sigma_n^2} H R_x H^* \right].$$

- This capacity is achieved by sending a Gaussian distributed signal with zeromean and covariance R_x.

- When the CSI is available at Tx, then the problem becomes finding the optimal input covariance R_x to maximize the capacity.

The result has a direct implication on the design of the transmitter and receiver communication strategies.

$$C = \max_{R_x} \log\det\left(I + \frac{1}{\sigma_n^2} H R_x H^* \right)$$
$$\text{s.t. } \operatorname{tr}(R_x) \leq P.$$

+) Parallel decomposition of a MIMO channel:

A MIMO matrix H can be decomposed into its singular value decomposition as

$$H = U \Sigma V^* \quad , H \in \mathbb{C}^{M_r \times M_t}$$

where

U : $M_r \times M_r$ (square) unitary matrix ($UU^* = U^*U = I$)
V : $M_t \times M_t$ (square) unitary matrix ($VV^* = V^*V = I$)
Σ : $M_r \times M_t$ diagonal matrix of singular values $\{\sigma_i\}$.

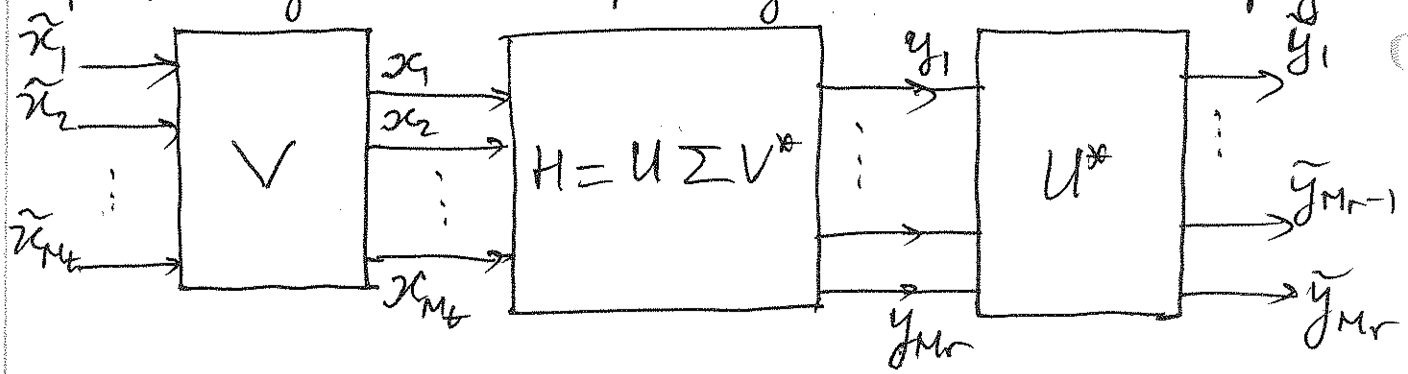
The singular values can be considered to be real. Both U and V are complex valued.

- The singular values are related to the eigenvalues of HH^* as:

$$\sigma_i = \sqrt{\lambda_i} \quad , \quad \begin{aligned} \lambda_i &= \text{eigenvalue of } HH^* \\ \sigma_i &= \text{singular value of } H. \end{aligned}$$

The number of non-zero singular values is equal to the rank of H.
If H is full rank then its rank is $r = \min\{M_r, M_t\}$.

- Parallel decomposition of a MIMO channel is obtained by performing transmit precoding and receiver shaping



$$y = Hx + n$$

then with receiver shaping and transmit precoding:

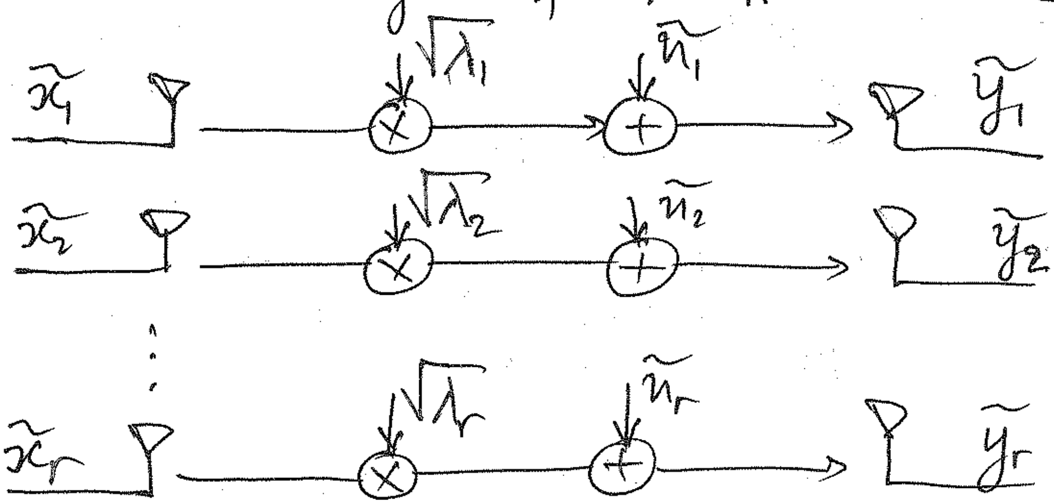
$$\begin{aligned} \tilde{y} = U^* y &= U^* Hx + U^* n \\ &= U^* H V\tilde{x} + \tilde{n} \\ &= U^* U \Sigma V^* V \tilde{x} + \tilde{n} \\ &= \Sigma \tilde{x} + \tilde{n} \end{aligned}$$

Thus the inputs and outputs in terms of $\tilde{x}$ and $\tilde{y}$ do not interfere with each other but act as parallel channels.

Note the new noise vector $\tilde{n}$ also contains independent components since

$$E[\tilde{n}\tilde{n}^*] = E[U^* n n^* U] = U^* E[n n^*] U$$
$$= \sigma_n^2 U^* U = I\sigma_n^2.$$

- Thus the MIMO channel is transformed into a set of parallel channels with gain of σ_i on each one.



$r = \text{rank}(H)$

- The power constraint transfers to the input $\tilde{x}$ as:

$$R_x = E[x x^*] = E[V\tilde{x}\tilde{x}^* V^*] = V \tilde{R}_x V^*$$

and since $tr(V\tilde{R}_x V^*) = tr(\tilde{R}_x)$

$$\rightarrow tr(\tilde{R}_x) = tr(R_x) \leq P.$$

Choose $\tilde{R}_x$ = diagonal as the capacity-optimal choice.

- The capacity optimization problem becomes a power allocation problem accross these parallel channels.

$$C = \max_{P_i} \sum_{i=1}^{r} \log\left(1 + \lambda_i \cdot \frac{P_i}{\sigma_n^2}\right)$$

$$\text{s.t.} \quad \sum_{i=1}^{r} P_i = P$$

P_i: power allocated to input $\tilde{x}_i$.

The solution is again a water-filling power allocation accross the sub channels:

$$P_i = \left(- \frac{P . \sigma_n^2}{\lambda_i} + \mu \right)^+ \qquad s.t. \sum_{i=1}^{r} P_i = P.$$

with μ such that $\sum P_i = P$.

- The fact that a MIMO channel can be decomposed into a number of parallel channels make the capacity increase linearly with the channel rank.

This linear increase in capacity is called the "multiplexing gain". The multiplexing gain is often the minimum between the number of Tx and Rx antennes.

- In fading channels, MIMO can bring both a multiplexing gain and a diversity gain.

- In a system with a single transmit (SIMO) or a single received antenna (MISO), no multiplexing gain is present (since the channel rank is 1), but diversity gain is possible.

2/ Capacity of fading MIMO channels with no CSIT:

- Without CSIT, the best thing the transmitter can do is to allocate equal power to all antennas without precoding.

- For fading, again we can look at the ergodic capacity and outage capacity.

+) <u>Ergodic capacity</u> (without CSIT):

$$C = \max E_H \left[\log\det \left(I + \frac{1}{\sigma_n^2} H R_x H^* \right) \right]$$

$$\text{s.t. } \operatorname{tr}(R_x) \le P.$$

where H is assumed to contain i.i.d entries that are zero_mean complex Gaussian (i.e. Rayleigh fading).

- It can be shown (Telatar 99) that the optimal input covariance matrix is

$$R_x = \frac{P}{M_t} \cdot I.$$

That is to send independent signals from all antennas with equal average power.

- The ergodic capacity is then

$$C = E_H \left[\log\det \left(I + \frac{P/\sigma_n^2}{M_t} H H^* \right) \right]$$

$$= E_{\lambda_i} \left[\sum_{i=1}^{r} \log \left(1 + \frac{P/\sigma_n^2}{M_t} \lambda_i \right) \right]$$

- It was show [Telatar 1999] that, quite surprisingly, even in this case of no CSIT, the capacity still grows linearly with the minimum between # of Tx & Rx antennas.

This result set off an excitement about the application of MIMO in wireless to increase capacity.

There were other results on outage capacity with MIMO [Foschini_Gans] that shows similar gain.
1998

- The ergodic capacity depends on the distribution of the eigenvalues of a random matrix and is quite complicated to analyze or evaluate.

We will look at a few special cases / asymptotic cases to get the intuition. But keep in mind that the linear capacity growth applies in most number of antennas configurations, even at a small number of antennas.

- For example: When $M_t \to \infty$ then $\frac{1}{M_t} HH^* \to I_{M_r}$ and the capacity approaches

$$C \to M_r \cdot \log\left(1 + \frac{P}{\sigma_n^2}\right)$$, linear in M_r.

- When both $M_t, M_r \to \infty$ but the ratio $\frac{M_r}{M_t}$ is fixed, then the distribution of the eigenvalues of HH^* approaches a fixed known distribution. Capacity can then be computed and shown to be linear in $M = \min(M_t, M_r)$. Thus linear growth holds for small M values as well.

- At high SNR:
The capacity grows linearly with $M = \min\{M_r, M_t\}$, which is called the multiplexing gain.

At v. low SNR, however, the capacity only grows with the number of receive antennas M_r, regardless of the number of transmit antennas M_t. At very low SNR, the benefit of multiple antennas is to collect the transmit energy coherently at the receiver, but it is not able to exploit the degree of freedom as in the high SNR case.