- The ergodic capacity depends on the distribution of the eigenvalues of a random matrix and is quite complicated to analyze or evaluate.

We will look at a few special cases / asymptotic cases to get the intuition. But keep in mind that the linear capacity growth applies in most number of antennas configurations, even at a small number of antennas.

- For example: When $M_t \to \infty$ then $\frac{1}{M_t} HH^* \to I_{M_r}$ and the capacity approaches

$$C \to M_r \cdot \log\left(1 + \frac{P}{\sigma_n^2}\right)$$, linear in M_r.

- When both $M_t, M_r \to \infty$ but the ratio $\frac{M_r}{M_t}$ is fixed, then the distribution of the eigenvalues of HH^* approaches a fixed known distribution. Capacity can then be computed and shown to be linear in $M = \min(M_t, M_r)$. This linear growth holds for small M values as well.

- At high SNR:
The capacity grows linearly with $M = \min\{M_r, M_t\}$, which is called the multiplexing gain.

- At v. low SNR, however, the capacity only grows with the number of receive antennas M_r, regardless of the number of transmit antennas M_t. At very low SNR, the benefit of multiple antennas is to collect the transmit energy coherently at the receiver, but it is not able to exploit the degree of freedom as in the high SNR case.

Using $\log_2(1+x) \approx x \log_2 e$ for small x, we can show that

$$C = \sum_{i=1}^{M} E \log\left(1 + \frac{\gamma}{M_t} \lambda_i^2\right) \approx \frac{\gamma}{M_t} \sum_{i=1}^{M} E \log(\lambda_i^2) = M_r \gamma \cdot \log_2 e \text{ (bps/Hz)}.$$

- For fading channels, the lack of CSIT does not affect the linear capacity growth of MIMO systems. But it complicates receiver design since we cannot decompose the channel into parallel channels. We will examine this topic later.

+) <u>Outage capacity</u>: $C = \log\det\left[I + \frac{P/\sigma_n^2}{M_t} H H^H\right]$
is a random quantity.

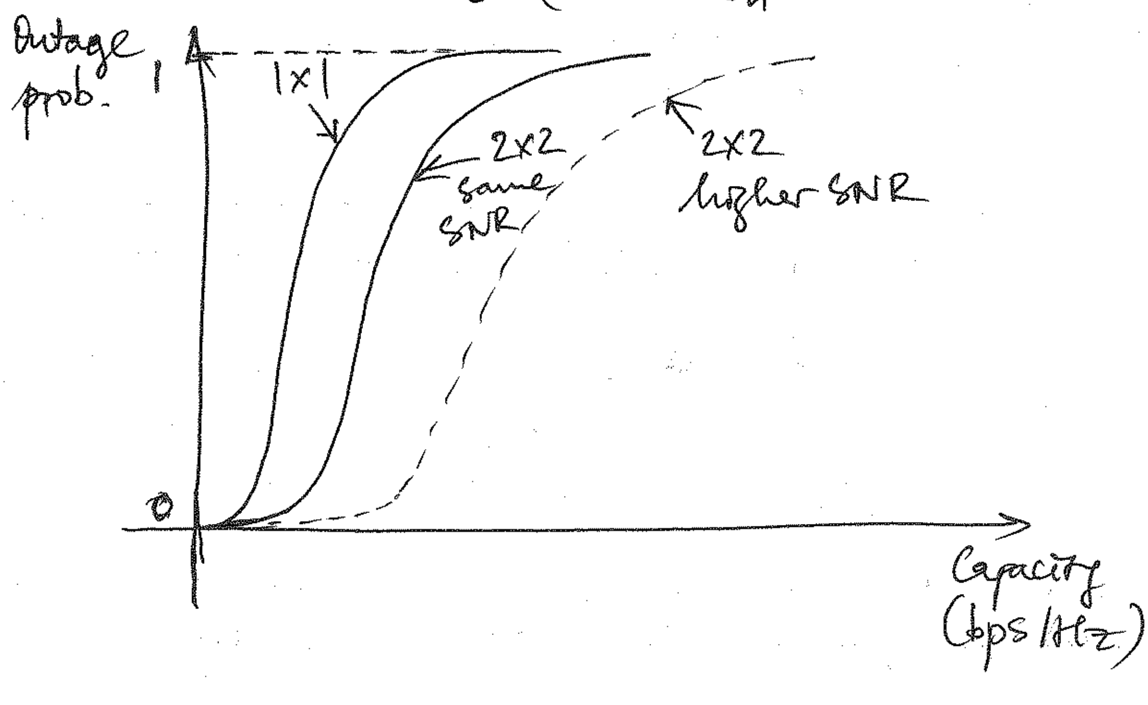
To obtain outage capacity, we need to understand the distribution of the capacity itself.

It can be shown that [Foschini_Gans 1998]

$$C > \sum_{k=M_t-(M_r-1)}^{M_t} \log_2\left[1 + \frac{P/\sigma_n^2}{M_t} \chi_{2k}^2\right], \quad M_t \geq M_r$$

This is to compare with the capacity of a single antenna system as

$$C_{siso} = \log_2\left(1 + \frac{P}{\sigma_n^2} \cdot \chi_2^2\right).$$



4/. Multiplexing and diversity trade-off in MIMO

+) We see that MIMO system provides a linear increase in capacity. This linear gain is referred to as the multiplexing gain.

. With CSIT:

$$C = \sum_{i=1}^{r} \log\left(1 + \frac{P_i}{\sigma_n^2} \cdot \lambda_i^2\right); \qquad r = \text{rank}(H), \quad \sum_{i=1}^{r} P_i = P$$

Without CSIT:

$$C = \sum_{i=1}^{r} E\left[\log\left(1 + \frac{P}{M_t \sigma_n^2} \cdot \lambda_i\right)\right] \qquad \left(P_i = \frac{P}{M_t} \ \forall i\right).$$

. The multiplexing gain in both cases is equal to r, the rank of the channel matrix.

For full-rank channel, then

$$r = \min(M_t, M_r)$$

A full-rank channel usually corresponds to a well scattered environment or well-spaced antennas to get independent channel coefficients within H.

. With CSIT, the number of multiplexing modes excited (used) depends on the SNR. (as the solution of water filling).

At very low SNR, we may just want to excite the largest eigenmode. In that case, MIMO is similar to receive diversity and provides a power and diversity gain.

As the SNR increases, we will want to use more eigenmodes for transmission until eventually all modes are excited.

The condition number of the channel gives an indication of how high an SNR has to be in order to use all multiplexing modes.

The condition number is defined as $\lambda_{max}/\lambda_{min}$. A channel is said to be well condition if this ratio is close to 1. A well conditioned channel will have all multiplexing modes excited at a lower SNR.

- Without CSIT, then all multiplexing modes are excited at all SNR.

- For $M_t \le M_r$, as the SNR increases, the capacity without CSIT approaches that with CSIT; thus the value of CSIT diminishes as SNR increases (in terms of capacity).

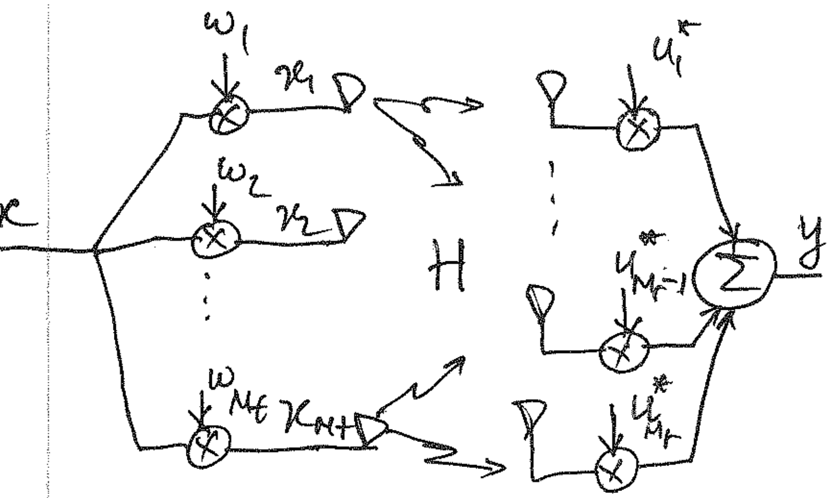
However, having CSIT can significantly simplify the transmitter and receiver design and is always valuable from that perspective.

- For $M_t > M_r$, having CSIT is important to avoid sending power into the null space of the channel (power sent into the null space will not reach the receiver).

+) MIMO diversity gain via beamforming and ST coding:

- Instead of capacity or multiplexing gain, multiple antennas in a MIMO system can be used to obtain diversity and array gain (as in diversity combining).

- Diversity is obtained through beamforming: Sending the same symbol weighted by different complex factors from all antennas.



$y = u^* H w x + u^* n$
where
$\|u\| = \|w\| = 1$

(the weight vectors are normalized to have norm 1 so not to increase power).

MIMO beamforming.
Diversity order $= M_t \times M_r$.
Only one data stream.

- This beamforming is usually performed when there is ~~both~~ CSIT and CSIR.
  Then u and w vector correspond to the first (priciple) left and right singular vectors of H (i.e. those corresponds to the largest singular value)

- MIMO beamforming is joint transmit and receive beamforming.

- MIMO beamforming achieves a diversity order of $M_t \times M_r$ and an array gain between $\max\{M_t, M_r\}$ and $M_t M_R$. Note the multiplexing gain here is only 1 since only one data stream is sent.

- When there is no CSIT, then to achieve diversity, space-time codes are used.

Space-time block codes can achieve a full diversity gain of $M_t \times M_r$, at a lower array gain than beamforming.

Space-time block codes (STBC) apply to flat fading channels. Space time trellis codes apply to frequency selective fading channels, but we will not go into the details here.

+) Multiplexing - diversity trade-off :

- A question arises is whether there exists a trade-off between multiplexing gain (number of independent data streams) and the diversity order in a MIMO system.

- When there is full CSIT and CSIR, then with singular value decomposition, and provided optimal channel code that achieves capacity, then full multiplexing gain and full diversity gain ($M_t \times M_t$) are both simultaneously possible.

- Even when there is only CSIR but no CSIT, full multiplexing gain of $\min(M_t, M_r)$ and full diversity gain of $M_t \times M_r$ is possible provided capacity-optimal code (Gaussian codebook with infinite block length). A scheme that can achieve this in theory is D-BLAST, we will cover later.

- When there is only CSIR (no CSIT) and given finite block code lengths, then it is not possible to simultaneously achieve the full diversity and full multiplexing gains simultaneously.

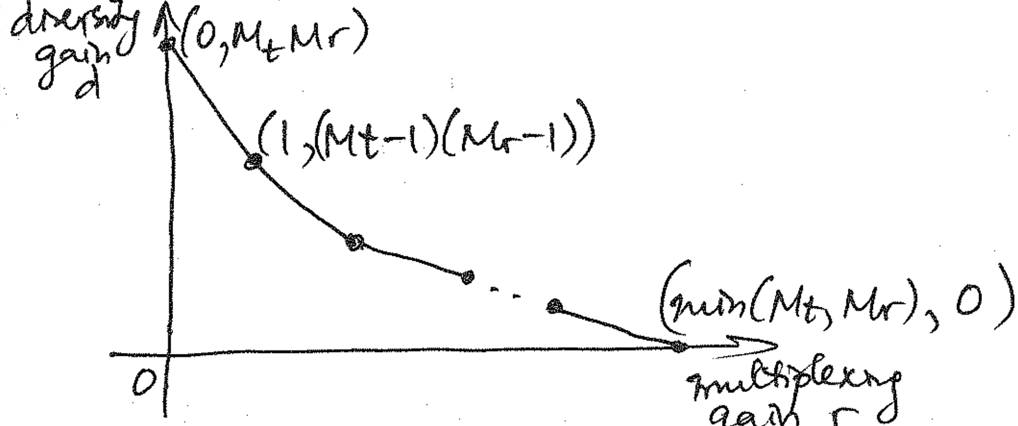
A characterization of multiplexing and diversity gains at high SNRs can be established as

Define:
$$\lim_{SNR \to \infty} \frac{R(SNR)}{\log_2 SNR} = r \triangleq \text{multiplexing gain}$$

$$\lim_{SNR \to \infty} \frac{\log P_e(SNR)}{\log SNR} = -d \triangleq \text{diversity gain}$$

Then for finite blocklength $T \geq M_r + M_t - 1$, the relation between the optimal diversity gain and multiplexing gain is

$$d_{opt}(r) = (M_t - r)(M_r - r) \quad , \quad 0 \leq r \leq \min(M_r, M_t)$$



It is possible to adapt the diversity and multiplexing gain according to channel state. If the channel is bad, more antennas can be used for diversity, if the channel is good, more antennas can be used for multiplexing.

5/ Space - time code design: rank and determinant criteria

- Consider a block fading channel.

Let X be the input space_time block code matrix

$$X = [x_1, \dots, x_T] \quad \text{of size } M_t \times T$$

each column x_i is sent from M_t antennas at time i.

Let Y be the received signal matrix over T symbols

$$Y = [y_1, \dots, y_T] \quad \text{of size } M_r \times T.$$