EE 193 - Applied Probability and Statistics for Engineers
Department of Electrical and Computer Engineering
Tufts University Fall 2007
Problem Set #3 Solutions

Problem 1

Yates and Goodman problems

- **2.2.3**

  (a) Since the sum of $P_V(v)$ over all v must equal 1 we have

$$\sum_{v=1}^{4} P_V(v) = c\left[1^2 + 2^2 + 3^2 + 4^2\right] = 1 \rightarrow 30c = 1 \rightarrow c = \frac{1}{30}$$

  (b) This is the probability that v is a square of an integer. Since v can assume values from 1 to 4, the only two squares are 1 and 4.

$$P[V \in \{u^2 | u = 1, 2, \dots\}] = P_V(1) + P_V(4) = \frac{1}{30} + \frac{16}{30} = \frac{17}{30}$$

  (c) $P[V \text{ is even}] = P[V = 2 \cup V = 4] = \frac{4}{30} + \frac{16}{30} = \frac{20}{30} = \frac{2}{3}$

  (d) $P[V > 2] = P[V = 3 \cup V = 4] = \frac{25}{30} = \frac{5}{6}$

- **2.2.6** Each call is a Bernoulli random variable with probability p. Failing to serve a person who calls three times means that person experiences three failures in a row. The probability of this occurring is $(1-p)^3$. Thus, the probability that the person is served is $1-(1-p)^3$ This must equal 0.95. Hence we require

$$1 - (1-p)^3 = 0.95 \rightarrow (1-p)^3 = 0.5 \rightarrow p = 1 - .05^{1/3} \approx 0.63$$

- **2.3.2** A Bernoulli process model is appropriate here where a "success" is the reception of a message on one trial. The probability of a success is p.

  (a) The probability we obtain K successes in n trials is a binomial random variable

$$P_K(k) = \binom{n}{k} p^k (1-p)^{n-k} k = 0, 1, 2, \dots$$

  (b) The probability of receiving the message at least once is 1 minus the probability of receiving the message zero times.

$$\begin{aligned} P[\text{At least once}] &= 1 - p_K(0) = 1 - \binom{n}{0}(0.8)^0(0.2)^n \\ &= 1 - 0.2^n \end{aligned}$$

  The requirement we have then is $1 - 0.2^n = 0.95$ or $0.2^n = .05$. Taking the natural log of both sides and dividing gives

$$\frac{\ln 0.05}{\ln 0.2} = 1.86$$

  Taking n to be the first integer larger than 1.86, that is 2, will give the required result.

- **2.3.11** The packet is transmitted once with probability p. For it to be transmitted twice, we need one failure and then one success for a probability of $(1-p)p$. For the packet to be transmitted three times, we need two failures followed by on success. This probability is $(1-p)^2 p$. As long as the number o tries is less than d, the probability is geometric. Now, if we have failed at time d then we stop transmitting. So the probability that we transmit d times is $(1-p)^d$. Thus it would appear that

$$p_T(t) = \begin{cases} p(1-p)^{t-1} & t = 0, 1, 2, \dots, d-1 \\ (1-p)^t & t = d \\ 0 & \text{else} \end{cases}$$

As a sanity check (and as an exercise in algebra), lets check that this sums to 1.

$$\begin{aligned} \sum_{t=1}^{d} p_T(t) &= p + p(1-p) + p(1-p)^2 + \dots p(1-p)^{d-1} + (1-p)^d \\ &= \left[\sum_{t=0}^{d-1} p(1-p)^t\right] + (1-p)^d = \left[p\frac{1-(1-p)^d}{1-(1-p)}\right] + (1-p)^d = 1 \end{aligned}$$

where in simplifying the sum in the second line, I have use the formula

$$\sum_{i=m}^{n} r^i = \frac{1-r^{n+1}}{1-r}$$

taking $r = 1-p$, $m = 0$ and $n = d-1$.

- **2.4.5** This is similar to problem 2.3.11. If the first pizza is mushroom, then with probability $p = 2/3$, there are zero pizzas sold before the first "success." If the first pizza is not mushroom but the second pizza is, then $N = 1$ with probability $(1-p)p$. More generally, the probability that N pizzas were sold before the first mushroom is equal to $(1-p)^N p$. This works for $0 \leq N < 99$. With probability $(1-p)^{100}$, there will be 100 pizzas sold before the first mushroom. Thus, the PMF is

$$p_N(n) = \begin{cases} p(1-p)^n & n = 0, 1, 2, \dots, 99 \\ (1-p)^{100} & n = 100 \\ 0 & \text{else} \end{cases}.$$

For $x < 0$, the CDF is zero. For integers between 0 and 100 the CDF is

$$F_N(n) = \sum_{i=0}^{n} p(1-p)^i = p\frac{1-(1-p)^{n+1}}{1-(1-p)} = 1 - \left(\frac{1}{3}\right)^{n+1}.$$

Thus using the "floor" notation from section 2.4 of the text, for $0 \leq x < 100$

$$F_N(x) = 1 - \left(\frac{1}{3}\right)^{\lfloor x \rfloor + 1}.$$

Finally for $x \geq 100$ the CDF is 1.

- **2.5.2**

  (a) From the question, a call will cost 20 cents (since it is voice) with probability 0.6 or it will cost 30 cents (since the call is data) with probability 0.4. Hence

  $$p_C(c) = \begin{cases} 0.6 & c = 20 \\ 0.4 & c = 30 \\ 0 & \text{else} \end{cases}$$

  (b) Using the definition of expected value: $E[C] = \sum_c c p_C(c) = 20 \times 0.6 + 30 \times 0.4 = 24$

- **2.5.9** The tree diagram for this problem is shown in Fig. 1. The top branches are the amount we take home after a win. The bottom branches show the progress of the game when we loose. All edges have probabilities equal to 1/2. As we see, we wither walk away with \$64 (regardless of how long it takes to win), or we loose everything and go home with \$0. The probability of loosing everything is $\left(\frac{1}{2}\right)^6 \approx 0.016$. The probability that we walk away with \$64 is $1 - \left(\frac{1}{2}\right)^6 \approx 0.984$. The expected value of the amount we take home is $64 \times 1 - (0.5)^6 = 62$. Since this the same as the amount we started with, this is not a game we would choose to play.

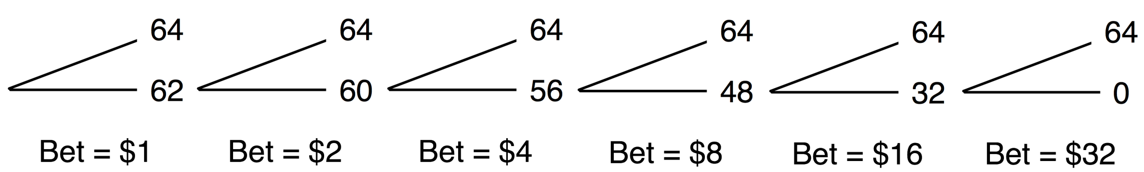

Figure 1: Problem 2.5.9

- **2.6.2** First, from the CDF of X in Problem 2.4.2, we find that the PDF is

  $$p_X(x) = \begin{cases} 0.2 & x = -1 \\ 0.5 & x = 0 \\ 0.3 & x = 1 \\ 0 & \text{else} \end{cases}.$$

  Now, since $V = |X|$, we have that $V = 1$ for $X = -1$ as well as $X = 1$ and $V = 0$ when $X = 0$. Hence $p_V(1) = p_X(-1) + p_X(1) = 0.5$ and $p_V(0) = p_X(0) = 0.5$ so

  $$p_V(v) = \begin{cases} 0.5 & v = 0 \\ 0.5 & v = 1 \\ 0.0 & \text{else} \end{cases}$$

  and

  $$F_V(x) = \begin{cases} 0 & x < 0 \\ 0.5 & 0 \leq x < 1 \\ 1 & x \geq 0 \end{cases}.$$

  Finally, $E[V] = 0 \times 0.5 + 1 \times 0.5 = 0.5$.

- **2.6.5**

  (a) Each transmission is a Bernoulli trial where the probability of error is q. The number of trials before the first non-error is thus geometric where, rather than waiting for the first "success" (here transmission with error) we are instead thinking about the first failure (no error)

  $$p_X(x) = q^{x-1}(1-q) \quad x = 1, 2, 3, \ldots$$

  (b) Let's start by enumerating what is going on. When $X = 1$ we are successful with the first attempt and $T = 1$. When $X = 2$, the sequence of events is transmit-NACK-transmit again so $T = 3$. Similarly, when $X = 3$ we have to go through 2 transmit-NACK cycles and then a fifth, successful transmit so $T = 5$. In general then $T = 2X - 1$ and $p_T(t) = p_X((t+1)/2)$ or

  $$p_T(t) = q^{(t+1)/2-1}(1-q) \quad t = 1, 3, 5, \ldots$$