EE 193 - Applied Probability and Statistics for Engineers
Department of Electrical and Computer Engineering
Tufts University Fall 2007
Problem Set #4: Solutions

Problem 1

Yates and Goodman problems

- **2.6.6** The relationship between cost C and minutes used M is

$$C = \begin{cases} 20 & 0 < M < 30 \\ 20 + \frac{1}{2}(M-30) & M \geq 30 \\ 0 & else \end{cases}$$

The PMF of M is

$$p_M(m) = \begin{cases} (1-p)^{m-1}p & m = 1, 2, 3, 4, \ldots \\ 0 & else \end{cases}$$

with $p = 1/30$. Now, if $M < 30$ the cost of the phone is \$20. So

$$p_C(20) = \sum_{m=1}^{29} (1-p)^{m-1}p = 1 - (1-p)^{30}$$

If M exceeds 30, then the cost goes up by \$0.50 per minute in excess of 30. We enumerate this as follows

$$\begin{aligned} p_C(20.5) &= p_M(31) = (1-p)^{30}p \\ p_C(21) &= p_M(32) = (1-p)^{31}p \\ p_C(21.5) &= p_M(33) = (1-p)^{32}p \\ &\vdots \end{aligned}$$

- **2.7.5** Repeating the beginning of the 2.6.6, the relationship between cost C and minutes used M is

$$C = \begin{cases} 20 & 0 < M < 30 \\ 20 + \frac{1}{2}(M-30) & M \geq 30 \\ 0 & else \end{cases}$$

From the PMF of the cost derived previously, the expected cost is

$$E[C] = 20 \times (1 - (1-p))^{30} + \sum_{m=31}^{\infty} \left(20 + \frac{1}{2}(m-30)\right)(1-p)^{m-1}p.$$

To do the sum, we change variables from m to $n = m - 30$. Hence we get

$$\begin{aligned} E[C] &= 20(1-(1-p))^{30} + \sum_{n=1}^{\infty}\left(20+\frac{1}{2}n\right)(1-p)^{n+30-1}p \\ &= 20(1-(1-p))^{30} + (1-p)^{30}\sum_{n=1}^{\infty}\left(20+\frac{1}{2}n\right)(1-p)^{n-1}p \\ &= 20(1-(1-p))^{30} + 20(1-p)^{30}\sum_{n=1}^{\infty}(1-p)^{n-1}p + \frac{1}{2}(1-p)^{30}\sum_{n=1}^{\infty}n(1-p)^{n-1}p \end{aligned}$$

The first sum is just one since we are adding up the probability for a geometric. The second sum is just the expected value of a geometric which is $1/p$. Hence,

$$E[C] = 20 - 20(1-p)^{30} + 20(1-p)^{30} + \frac{(1-p)^{30}}{2p} = 20 + \frac{(1-p)^{30}}{2p}$$

- **2.8.9** From problem 2.6.5, the interarrival time for a packet is related to the number of times the packet is transmitted by the source via $T = 2X - 1$. Hence, $\sigma^2 T = 4\sigma^2 X$ or $\sigma_T = 2\sigma_X$. Since X is geometric, we get from the appendix

  $$\sigma_T = \frac{2(1-q)}{q^2}.$$

  To meet the required jitter specification, we need

  $$\frac{2(1-q)}{q^2} < 2 \rightarrow q^2 + q - 1 > 0.$$

  Solving the quadratic for q gives a single positive root at $(\sqrt{5}-1)/2$.

- **2.9.7**

  (a) The tree diagram for this problem is shown in Fig. 1. The labels at the end of each branch correspond to the number of miles run that day. We see from this figure that

  $$\begin{aligned} p_M(0) &= q \\ p_M(1) &= (1-q)q \\ p_M(2) &= (1-q)^2 q \\ &\vdots \\ p_M(m) &= (1-q)^m q \quad m = 0, 1, 2, \ldots \end{aligned}$$

  and that $P[M > 0] = 1 - P[M = 0] = 1 - q$.

  (b) The probability that we run a marathon is just the probability that M is greater than

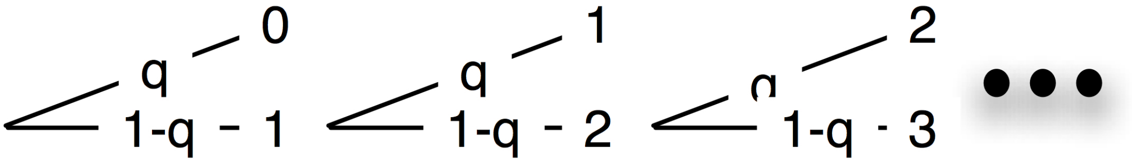


Figure 1: Problem 2.9.7(a)

or equal to 26 which can be computed as

$$\begin{aligned} r &= P[M \geq 26] = \sum_{m=26}^{\infty} (1-q)^m q \\ &= q(1-q)^{26} + q(1-q)^{27} + q(1-q)^{28} + \dots \\ &= q(1-q)^{26}\left(1 + (1-q) + (1-q)^2 + \dots\right) = q(1-q)^{26} \sum_{m=0}^{\infty} (1-q)^m \\ &= q(1-q)^{26} \frac{1}{1-(1-q)} = (1-q)^{26} \end{aligned}$$

(c) Each day, we run a marathon with probability r. Thus we have a Bernoulli trial each day where a "success" is interpreted as a marathon run. Thus the number of marathons in a year is the number of successes in 365 trials which is just a binomial:

$$p_J(j) = \binom{365}{j} r^j (1-r)^{365-j} \quad j = 0, 1, 2, \dots, 365$$

(d) From the definition of conditional probability we have

$$p_K|A(k|A) = \frac{P[K = k \cap A]}{P[A]}.$$

Now, the event A means that we have run at least 26 miles. Note that this has been computed in (b). Thus $M \geq 26$ which means $K = M - 26$ will take on values 0, 1, 2, $\dots$. Now, $P[K = 0 \cap A]$ is the probability that $M = 26$ which from (a) is $(1-q)^{26}q$. Similarly, $P[K = 1 \cap A]$ is the probability that $M = 27$ which from (a) is $(1-q)^{27}q$. In general $P[K = k \cap A] = p_M(26 + k) = (1-q)^{26+k}q$. Hence

$$p_K|A(k|A) = \frac{P[K = k \cap A]}{P[A]} = \frac{(1-q)^{26+k}q}{(1-q)^{26}} = (1-q)^k q \quad k = 0, 1, 2, \dots.$$

- **2.10.5** The results are shown in Fig. 2 aand the code is:

```
% The number of samples we are going to use.
n = [100 ; 1000; 10000];
% Loop over each
for idx = 1:3
  % Generate the random deviates
```

```
  y = poissrnd(5*ones(n(idx),1));

  % The max of y is how far out we need to go in terms of
  % estimatng the PDF
  my = max(y);x = (0:my)';
  % For loop to build up the reltive frequency information
  ny = zeros(my+1,1);
  for nidx = 0:my
    ny(nidx+1) = sum(y==nidx)/n(idx);
  end
  % Generate the "theoretical" values for the PDF
  p = poisspdf(x,5);
  % Plot and print the results
  plot(x,p,'-',x,ny,'--');
  xlabel('n')
  title(['n = ',num2str(n(idx))]);
  legend('Theory','Sample');
  pstr = ['print -djpeg p2p10p5_',num2str(idx),'.jpg'];
  eval(pstr);
end
```

We get reasonable agreement with theory for $n = 1000$.

- **3.2.2** The PDF of $F_X(x)$ is $f_X(x) = dF_X/dx$ which in this case is

$$f_X(x) = \begin{cases} 0 & x < 1 \\ 1/2 & -1 \le x < 1 \\ 0 & x \ge 1 \end{cases}$$

- **3.2.4** From the definition of the CDF we have for $x > 0$

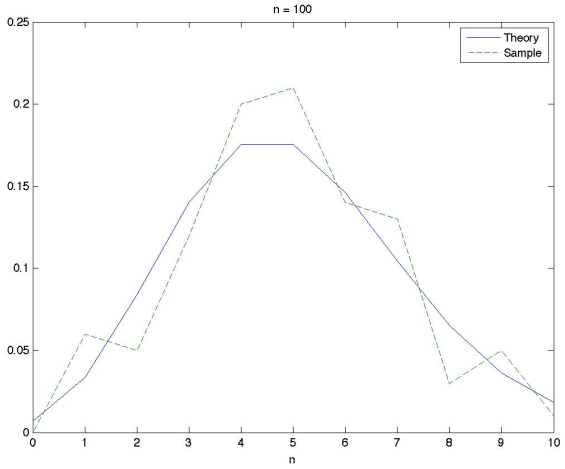
$$F_X(x) = \int_{-\infty}^{x} a^2 y e^{-a^2y^2/2}$$

and $F_X(x) = 0$ for $x \le 0$. To do the integral, we make the change of variable $q = y^2$ so that:

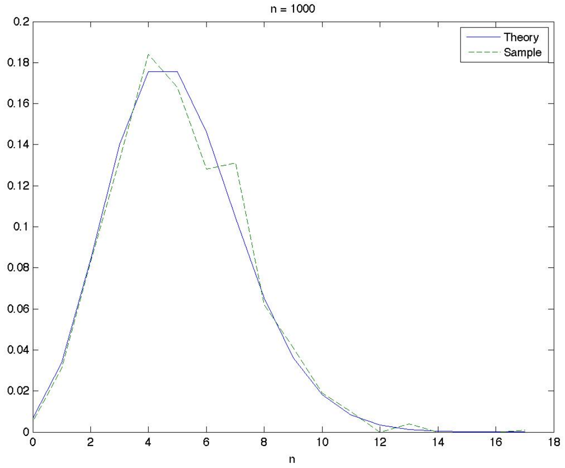
  - $dq = 2ydy \rightarrow \frac{1}{2}dq = ydy$
  - $y = -\infty \rightarrow q = +\infty$
  - $y = x \rightarrow q = x^2$

Thus the integral is

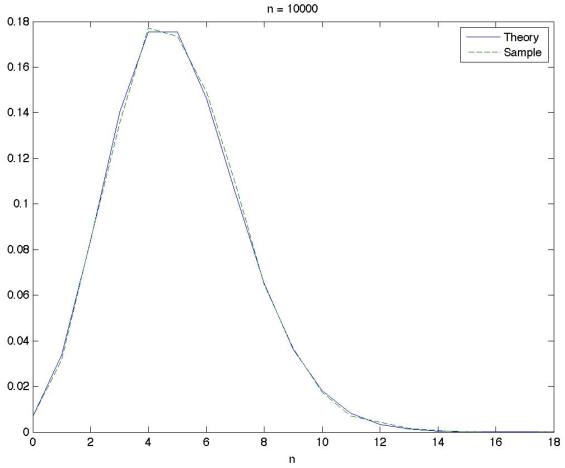
$$\begin{aligned} F_X(x) &= \frac{a^2}{2}\int_{\infty}^{x^2} e^{-a^2q/2}dq = -\frac{a^2}{2}\int_{x^2}^{\infty} e^{-a^2q/2}dq \\ &= -\frac{a^2}{2}\frac{2}{a^2}e^{a^2q/2}\Big|_{x^2}^{\infty} = -1\left[0 - e^{a^2x^2/2}\right] \\ &= e^{a^2x^2/2} \end{aligned}$$



(a) $n = 100$



(b) $n = 1,000$



(c) $n = 10,000$

Figure 2: Poisson simulation results

- **3.3.2**

  (a) From the table at the end of the text, $E[X] = (9+1)/2 = 5$ and $\text{var}[X] = (9-1)^2/12 = 16/3$

  (b) First $h(E[X]) = 1/\sqrt{E[X]} = 1/\sqrt{5}$. Second,

$$\begin{aligned} E[h(X)] &= \int_1^9 \frac{1}{\sqrt{x}}\frac{1}{8}dx \\ &= \frac{1}{8}\frac{1}{1/2}x^{1/2}\bigg|_1^9 = \frac{1}{4}\left[\sqrt{9}-1\right] = \frac{1}{2} \neq \frac{16}{3} \end{aligned}$$

  (c) We found $E[Y] = E[h(X)]$ in part (b). To find the variance, we compute the second moment:

$$\begin{aligned} E[Y^2] &= E[\frac{1}{X}] = \int_1^9 \frac{1}{8}\frac{1}{x}dx \\ &= \frac{1}{8}\left(\ln(9) - \ln(1)\right) = \frac{\ln 9}{8} \end{aligned}$$

  So, $\text{var}[Y] = \ln(9)/8 - 1/4$

- **3.3.6** For this problem, let's compute the PDF and then the first three moments. The PDF is

$$f_V v = \frac{dF_V}{dv} = \begin{cases} 0 & v < -5 \text{ and } v \geq 7 \\ \frac{v+5}{72} & -5 \leq v < 7 \end{cases}$$

  Now the moments:

$$\begin{aligned} E[V] &= \int_{-5}^7 \frac{1}{72}\left(v^2+5v\right) = \frac{1}{72}\left(\frac{1}{3}v^3 + \frac{5}{2}v^2\right)_{-5}^7 \\ &= \frac{1}{72}\left(\frac{343}{3} + \frac{245}{3} + \frac{125}{3} - \frac{125}{3}\right) = 3 \\ E[V^2] &= \int_{-5}^7 \frac{1}{72}\left(v^3+5v^2\right) = \frac{1}{72}\left(\frac{1}{4}v^4 + \frac{5}{3}v^3\right)_{-5}^7 = \frac{6719}{432} \approx 15.55 \\ E[V^3] &= \int_{-5}^7 \frac{1}{72}\left(v^4+5v^3\right) = \frac{1}{72}\left(\frac{1}{5}v^5 + \frac{5}{4}v^4\right)_{-5}^7 \approx 86.2 \end{aligned}$$

  So, we have answered parts (a) and (c). The variance is just $E[V^2]-(E[V])^2 \approx 15.55-9 = 6.55$