EE 193 - Applied Probability and Statistics for Engineers
Department of Electrical and Computer Engineering
Tufts University Fall 2007
Problem Set #4 Solutions

Reading: Yates and Goodman, Chapter 3

Problem 1

- **3.4.2** From the Appendix in the text, the variance of an exponential RV is $1/\lambda^2$. Since the variance of Y is 25, we conclude that for this RV, $\lambda = 1/5$.
  - The PDF is
  $$p_Y(y) = \begin{cases} \frac{1}{5}e^{y/5} & y > 0 \\ 0 & \text{else} \end{cases}$$
  - Since the variance satisfies $\sigma_Y^2 = E[Y^2] - (E[Y])^2$ we have $E[Y^2] = \sigma_Y^2 + (E[Y])^2 = 1/\lambda^2 + 1/\lambda^2 = 2/\lambda^2 = 2/25$
  - The required probability is
  $$P[Y > 5] = \int_5^\infty \frac{1}{5}e^{-y/5}dy = e^{-1}$$

- **3.4.9**Let T be the time for a call. In terms of T, the costs for plans A and B are:
$$\begin{aligned} C_A &= 10T \\ C_B &= \begin{cases} 99 & T < 20 \\ 99 + 10(T-20) & T > 20 \end{cases} \end{aligned}$$
Now we computed expected values:
$$\begin{aligned} E[C_A] &= \int_0^\infty \frac{1}{\tau}10te^{-t/\tau}dt \\ &= 10E[T] = 10\tau \\ E[C_B] &= \int_0^{20} 99\frac{1}{\tau}e^{-t/\tau}dt + \int_{20}^\infty (99 - 10(t-20))\frac{1}{\tau}e^{-t/\tau}dt \\ &= 99 - 10\tau e^{-20/\tau} \end{aligned}$$
Playing around a bit with Matlab, gives us that for τ greater than about 12.34 minutes, calling plan B is a better deal than calling plan A.

- **3.5.3** We have that $A \sim (N(0, \sigma_X)$ meaning
$$P[|X| \le 10] = P\left[\frac{|X|}{\sigma_X} \le \frac{10}{\sigma_X}\right] = \Phi\left(\frac{10}{\sigma_X}\right) - \Phi\left(-\frac{10}{\sigma_X}\right) = 2\Phi\left(\frac{10}{\sigma_X}\right) - 1 = 0.1$$
where I have used the method of Example 3.17. Now, solving the above for σ_X gives
$$\sigma_X = 10\left[\Phi^{-1}\left(\frac{1.1}{2}\right)\right]^{-1} \approx 10/0.13 \approx 77$$

- **3.5.5** We have $T \sim N(\mu, 15)$. From the problem statement we know

$$P[T > 10] = 0.5 \rightarrow P\left[\frac{t-\mu}{15} > \frac{10-\mu}{15}\right] = 1 - \Phi\left(\frac{10-\mu}{15}\right) = \frac{1}{2}$$

which means $\Phi((10-\mu)/15) = 1/2$. Since $\Phi^{-1}(0.5) = 0$ we conclude that $\mu = 10$. Hence

$$\begin{aligned} P[T > 32] &= P\left[\frac{T-10}{15} > \frac{32-10}{15}\right] \\ &= 1 - \Phi(22/15) = 1 - \Phi(1.47) \approx 1 - 0.9292 = .0708 \\ P[T < 0] &= P\left[\frac{T-10}{15} < \frac{0-10}{15}\right] \\ &= \Phi(-0.66) = 1 - \Phi(0.66) \approx 1 - 0.7454 = 0.2546 \\ P[T > 60] &= P\left[\frac{T-10}{15} < \frac{60-10}{15}\right] \\ &= \Phi(3.33) = 0.9996 \text{ using } \texttt{normcdf} \text{ in Matlab} \end{aligned}$$

- **3.5.7** Using inches as the standard unit of height in the problem, we let the RV $X \sim N(70, \sigma_X)$ be the height of a U.S. male. We interpret the statement that 23,000 of 100,000,000 males are over 7 feet (84 inches) tall to mean

$$P[X > 84] = \frac{23,000}{100,000,000}.$$

(a) Standardizing gives

$$P\left[\frac{X-70}{\sigma_X} > \frac{84-70}{\sigma_X}\right] = 1 - \Phi\left[\frac{14}{\sigma_X}\right] = \frac{23}{100,000}.$$

Solving for σ_X gives

$$\sigma_X = \frac{14}{\Phi^{-1}\left(1 - \frac{23}{100,000}\right)} \approx 4.00 \text{ using } \texttt{norminv} \text{ in Matlab}$$

(b) We have

$$P[X > 96] = P\left[\frac{X-70}{4} > \frac{96-70}{4}\right] = 1 - \Phi(6.5)$$

(c) The probability that any individual is over 90 inches is

$$P[X > 90] = 1 - \Phi\left(\frac{90-70}{4}\right) = 1 - \Phi(5) \approx 2.87e-7 \equiv p.$$

Now, for this problem, there are 100,000,000 men in the U.S. If each man is a Bernoulli trial with $p = 2.87e-7$, then the probability that there are no men over 90 inches in height is $(1-p)^{100{,}000{,}000} \approx 3.56e-13$. This is a very small number indeed.

(d) Here we are asking for the expected number of "successes" in 100,000,000 trials of a binomial RV with $p = 2.87e-7$. This is just $100,000,000 \times p \approx 28.7$ or roughly 30 men.

- **3.7.2** We start by finding the CDF for Y:

$$\begin{aligned} F_Y(y) = P[Y < y] &= P[\sqrt{X} < y] = P[X < y^2] \\ &= \int_0^{y^2} \lambda e^{-\lambda x} dx \\ &= 1 - e^{-\lambda y^2} \end{aligned}$$

which holds for $y \geq 0$. Differentiating with repsect to y gives the PDF:

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 2\lambda y e^{-\lambda y^2} & y \geq 0 \\ 0 & \text{else} \end{cases}$$

The form of a Rayleigh distribution is

$$f_X(x) = \begin{cases} a^2 x e^{-x^2 a^2/2} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

The two are the same if $a^2 = 2\lambda$ or $a = \sqrt{2\lambda}$.

- **3.7.8**

  (a) First, the radius, R, of a circle with unit circumference satisfies $2\pi R = 1 \to R = 1/2\pi$. Now, if we let A denote the angle subtended by X we know from elementary analytic geometry that $A = 2\pi X$. Next, Y, the area of the circle with radius $1/2\pi$ enclosed in a sector with angle A is, using integration in polar coordinates:

$$Y = \int_0^{1/2\pi} r dr \int_0^A d\theta = \frac{A}{8\pi^2}$$

  but since $A = 2\pi X$ we end up with

$$Y = \frac{X}{4\pi}$$

  (b) Using the definition of $F_Y(y)$ we have

$$\begin{aligned} F_Y(y) &= P[Y < y] = P\left[\frac{X}{4\pi} < y\right] \\ &= P[X < 4\pi y] = \begin{cases} 0 & y < 0 \\ \int_0^{4\pi y} 1 dx = 4\pi y & 0 < y < \frac{1}{4\pi} \\ 1 & y > \frac{1}{4\pi} \end{cases} \end{aligned}$$

  (c) Taking derivatives we get

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 4\pi & 0 < y < \frac{1}{4\pi} \\ 0 & \text{else} \end{cases}$$

(d) This is a uniform RV with parameters $a = 0$ and $b = \frac{1}{4\pi}$. Hence the expected value is $\frac{1}{8\pi}$.

- **3.8.1**

(a) The probability of B is

$$P[B] = \int_{-3}^{3} \frac{1}{10} dX = 6/10.$$

So the conditional PDF is

$$f_X|B(x|B) = \frac{f_X(x)}{P[B]} \quad x \in B = \begin{cases} \frac{1/10}{6/10} = \frac{1}{6} & |x| < 3 \\ 0 & \text{else} \end{cases}$$

(b) Since the conditional PDF is uniform with $a = -3$ and $b = 3$, the conditional expected value is $E[X|B] = (a + b)/2 = 0$

(c) From the a and b parameters identified in part (b), $\sigma^2_{X|B} = (b - a)^2/12 = 36/12 = 3$

- **3.8.5** Since $E[T] = .01 = 1/\lambda$, the PDF for T is

$$f_T(t) = \begin{cases} 100e^{-100t} & t > 0 \\ 0 & \text{else} \end{cases}$$

and the probability of the event $A = \{T > .02\}$ is

$$P[T > .02] = \int_{.02}^{\infty} 100e^{-100t} dt = e^{-2}.$$

Hence, the conditional PDF is

$$f_{T|A}(t|A) = \begin{cases} 100e^{-2}e^{-100t} = 100e^{-100(t-.02)} & t > 0.02 \\ 0 & \text{else} \end{cases}$$

But this is just a “shifted” exponential random variable. Hence, it’s mean (i.e., the conditional expected value) will be the mean of the original,0.01, plus 0.02 which is 0.03. Since we still have a decay paramater, $\lambda = 100$ (i.e., since the shape of the random variance is the same regardless of the shift along the T axis) the conditional variance will be the same as the original, $1/\lambda^2 = 10^{-4}$.

- **3.8.7**

(a) Given that a person is healthy, the PDF is Gaussian with mean 90 and standard deviation 20. Hence the PDF is

$$f_{X|H}(x|H) = \frac{1}{\sqrt{800\pi}} e^{-(x-90)/800}$$

(b)

$$\begin{aligned} P[T^+|H] &= P[X \geq 140|H] = P\left[\frac{X - 90}{20} \geq \frac{140 - 90}{20}\right] = 1 - \Phi(5/2) = 0.006 \\ P[T^-|H] &= P[X \leq 110|H] = P\left[\frac{X - 90}{20} \leq \frac{110 - 90}{20}\right] = \Phi(1) = 0.841 \end{aligned}$$

(c) Start from the definition of conditional probability

$$P[H|T^-] = \frac{P[HT^-]}{P[T^-]} = \frac{P[T^-|H]P[H]}{P[T^-|H]P[H] + P[T^-|D]P[D]}.$$

We have all of the pieces of the formula on the right hand side except for $P[T^-|D]$ which is

$$P[T^-|D] = P[X \leq 110|D] = P\left[\frac{X-160}{40} \leq \frac{110-160}{40}\right] = 1 - \Phi(5/4) = 0.106.$$

Putting everything together gives $P[H|T^-] = 0.986$.

(d) Note that $P[T^0|H] = 1 - P[T^+|H] - P[T^-|H] = 0.153$. We can think of this need to repeat tests as a Bernoulli process where a "failure" is given my an ambiguous test and has probability of $p = 0.153$ while a success is an unambiguous result and occurs with probability $1-p = 0.847$. The number number of tests until the first unambiguous result given that a person is healthy is a geometric random variable

$$p_{N|H}(n|H) = \begin{cases} (1-p)p^n & n = 1, 2, 3, \ldots \\ 0 & \text{else} \end{cases}$$