

EE 193 - Applied Probability and Statistics for Engineers
Department of Electrical and Computer Engineering
Tufts University Fall 2007
Problem Set #9
Distributed Nov. 29,2007
Due Never

Problem 1

Yates and Goodman problems

- **8.2.2 (a)–(e)** Here we have a binary hypothesis testing problem with:

$$H_0 : \quad t \sim f_V(t) = \begin{cases} \frac{1}{3}e^{-t/3} & t \geq 0 \\ 0 & \text{else} \end{cases}$$
$$H_1 : \quad t \sim f_D(t) = \begin{cases} \frac{1}{\mu_D}e^{-t/\mu_D} & t \geq 0 \\ 0 & \text{else} \end{cases}$$

We also know $P[H_0] = 0.8$ and $P[H_1] = 0.2$. The graphs for this problem are illustrated in Figure

- (a) The false alarm probability is the probability we declare H_1 given that H_0 is correct. Hence, this is the area under the H_0 curve to the right of t_0 :

$$P_{FA} = \int_{t_0}^{\infty} \frac{1}{3}e^{-t/3} dt = e^{-t_0/3}$$

- (b) The miss probability is the probability that we declare H_0 when H_1 is true. This then is the area under the H_1 curve for $0 \leq t \leq t_0$:

$$PM = \int_0^{t_0} \frac{1}{\mu_D}e^{-t/\mu_D} dt = 1 - e^{-t_0/\mu_D}$$

- (c) The likelihood ratio test (LRT) for this problem is

$$\frac{\frac{1}{3}e^{-t/3}}{\frac{1}{\mu_D}e^{-t/\mu_D}} \geq 1 \quad \text{declare } H_0 \text{ else declare } H_1$$

We can rearrange the above inequality with a bit of algebra to get

$$t \leq \left[\frac{1}{3} - \frac{1}{\mu_D} \right]^{-1} \ln \left(\frac{\mu_D}{3} \right) \quad \text{declare } H_0 \text{ else declare } H_1$$

For $\mu_D = 6$ minutes, the threshold is about 4.16 minutes while for $\mu_D = 10$ minutes, the threshold is about 5.16 minutes.

- (d) We would expect the MAP threshold to be higher since we know that the probability of a voice call is four times more likely that of a data call.

(e) Proceeding as in part (c), the likelihood ratio test for the MAP case is

$$\frac{\frac{1}{3}e^{-t/3}}{\frac{1}{\mu_D}e^{-t/\mu_D}} \geq \frac{0.2}{0.8} \quad \text{declare } H_0 \text{ else declare } H_1$$

We can rearrange the above inequality with a bit of algebra to get

$$t \leq \left[\frac{1}{3} - \frac{1}{\mu_D} \right]^{-1} \ln \left(\frac{4\mu_D}{3} \right) \quad \text{declare } H_0 \text{ else declare } H_1$$

For $\mu_D = 6$ minutes, the threshold is about 12.5 minutes while for $\mu_D = 10$ minutes, the threshold is about 11.1 minutes.

• **8.2.4**

(a) Under H_0 , $X \sim N(0, 1)$ while under H_1 , $X \sim N(4, 1)$. The MAP LRT for this problem is

$$\frac{\frac{1}{\sqrt{2\pi}}e^{x^2/2}}{\frac{1}{\sqrt{2\pi}}e^{(x-4)^2/2}} \geq \frac{0.01}{0.99} \quad \text{declare } H_0 \text{ else declare } H_1$$

Rearranging gives

$$x \leq x_{MAP} = 2 - \frac{1}{4} \ln(.01/.99) \approx 3.15 \quad \text{declare } H_0 \text{ else declare } H_1$$

The false alarm probability is

$$\begin{aligned} P_{FA} &= P[X > X_{MAP} | H_0] = P \left[\frac{X - \mu_{X|H_0}}{\sigma_{X|H_0}} > \frac{X_{MAP} - \mu_{X|H_0}}{\sigma_{X|H_0}} \middle| H_0 \right] \\ &= 1 - \Phi(X_{MAP}) \approx 8.16 \times 10^{-4} \end{aligned}$$

where we have made use of the fact that $\mu_{X|H_0} = 1$ and $\sigma_{X|H_0} = 1$.

The miss probability is

$$\begin{aligned} P_M &= P[X < X_{MAP} | H_1] = P \left[\frac{X - \mu_{X|H_1}}{\sigma_{X|H_1}} < \frac{X_{MAP} - \mu_{X|H_1}}{\sigma_{X|H_1}} \middle| H_1 \right] \\ &= \Phi(X_{MAP} - 4) \approx 0.1977 \end{aligned}$$

Finally, for the MAP problem, the average cost is

$$E[C] = C_{10}P_{FA}P[H_0] + C_{01}P_MP[H_1].$$

For a MAP test, $C_{01} = C_{10} = 1$. The remaining components of $E[C]$ are known and we get $E[C] = 19.77$.

(b) For the min cost test, C_{10} stays at 1 while $C_{01} = 10^4$. The LRT is now

$$\frac{\frac{1}{\sqrt{2\pi}}e^{x^2/2}}{\frac{1}{\sqrt{2\pi}}e^{(x-4)^2/2}} \geq \frac{10^4 \times 0.01}{0.99} \quad \text{declare } H_0 \text{ else declare } H_1$$

Rearranging gives

$$x \leq x_{MC} = 2 - \frac{1}{4} \ln(10^4 \times .01/.99) \approx 0.846 \quad \text{declare } H_0 \text{ else declare } H_1$$

The error probabilities are now

$$P_{FA} = 1 - \Phi(x_{MC}) \approx 0.1987 \quad (1.1.1)$$

$$P_M = \Phi(x_{MC} - 4) \approx 8.06 \times 10^{-4} \quad (1.1.2)$$

and the executed cost is

$$E[C] = C_{10}P_{FA}P[H_0] + C_{10}P_MP[H_1] = 0.2773.$$

Because the cost of a miss is so high, the min cost test really drives down the miss probability thereby reducing the overall cost significantly.

- **8.2.8** For this problem, we start by observing that when X is zero, $Y = W$. Since W is exponential, we have

$$f_{Y|X}(Y|X = 0) = \begin{cases} e^{-y} & y \geq 0 \\ 0 & \text{else} \end{cases}$$

Next, when $X = 1$, we have $Y = V + W$. Now, V and W are both IID exponential random variables. Hence their sum is an Erlang (see Theorem 6.11 on page 253) and

$$f_{Y|X}(Y|X = 1) = \begin{cases} ye^{-y} & y \geq 0 \\ 0 & \text{else} \end{cases}$$

The two conditional PDFs for this problem are shown in Fig. 2. Following the discussion in example 8.6 of the text, the minimum probability of error decision is

$$\frac{f_{Y|X}(Y|X = 0)}{f_{Y|X}(Y|X = 1)} = \frac{e^{-y}}{ye^{-y}} \geq \frac{1-p}{p} \quad \text{declare } H_0 \text{ else declare } H_1.$$

Since $p = 1/2$ for this problem, the rule reduces to

$$y \leq 1 \quad \text{declare } H_0 \text{ else declare } H_1.$$

Following the reasoning in Example 8.7 of the text on page 307

$$P_{ERR} = P[Y > 1|H_0]P[X = 0] + P[Y < 1|H_1]P[X = 1].$$

Now, $P[X = 0] = P[X = 1] = 0.5$ and

$$P[Y > 1|H_0] = \int_1^\infty e^{-y} dy = e^{-1}$$

$$P[Y < 1|H_1] = \int_0^1 ye^{-y} dy = 1 - 2e^{-1}$$

Hence $P_{ERR} = \frac{1}{2}(1 - e^{-1})$.

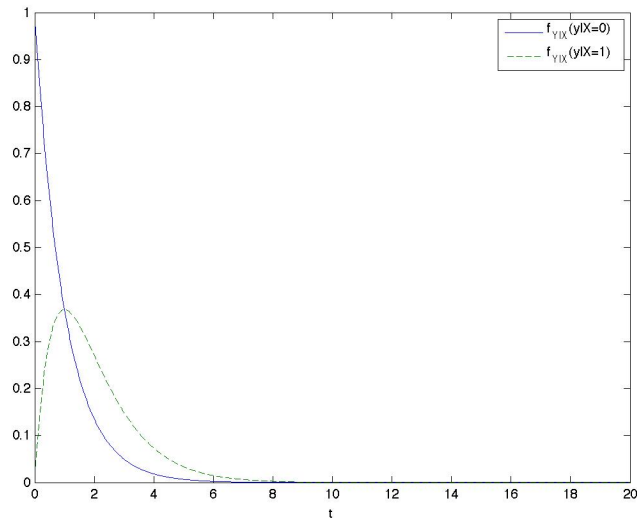


Figure 1: Problem 8.2.2

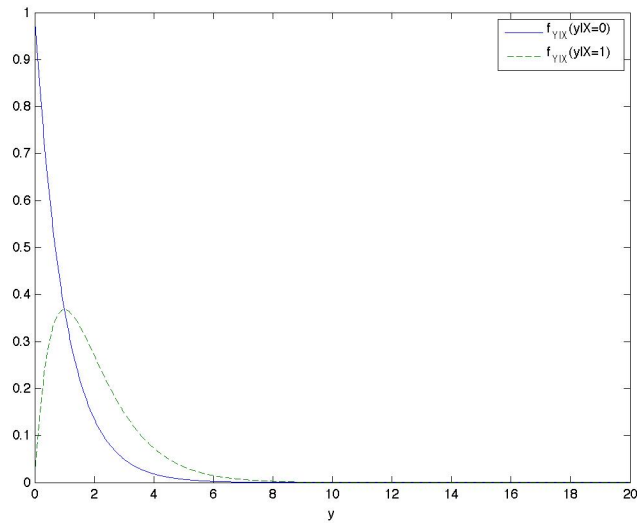


Figure 2: Problem 8.2.8