Example: \( \min f_0(x) = -\sum_{i=1}^{k} \log(a_i^T x + b_i) \)
is an unconstrained problem with implicit constraints as
\( a_i^T x - b_i \leq 0 \).

1) Feasibility problem: This kind of problems is to determine if a set of constraints is feasible (or consistent).

A feasibility problem can be written as
\[
\begin{align*}
\text{find } x & & \min & 0 \\
\text{s.t. } f_i(x) \leq 0, & & i = 1, \ldots, m \text{ OR } \\
h_i(x) = 0, & & i = 1, \ldots, p
\end{align*}
\]

2. Convex optimization problems.

\[
\begin{align*}
\min & \quad f_0(x) \\
\text{s.t. } & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad a_i^T x = b_i, \quad i = 1, \ldots, p
\end{align*}
\]

where \( f_0, f_1, \ldots, f_m \) are convex functions.
the equality constraints are affine.

1) If \( f_0(x) \) is quasiconvex, the problem is quasiconvex.
(other \( f_i(x) \) are still convex).

1) The feasible set of a convex optimization problem is convex,
\[
D = \bigcap_{i=0}^{m} \text{dom} f_i \cap \bigcap_{i=1}^{p} \text{dom} h_i
\]

sublevelsets \( \{x | f_i(x) \leq 0 \} \)
hyperplanes \( \{x | a_i^T x = b_i \} \)
Note that this definition of convex problem is strict. A more general definition, or abstract convex problem, is that minimizes a convex function over a convex set. As it turns out, solving such an abstract convex optimization problem requires us to find a description of the set in terms of convex inequalities and linear equality constraints. For this reason, we focus on this restricted form of convex problems.

Example: \[
\begin{align*}
\text{min} & \quad x_1 + x_2 \\
\text{s.t.} & \quad -x_1 \leq 0 \\
 & \quad -x_2 \leq 0 \\
 & \quad 1 - x_1 x_2 \leq 0.
\end{align*}
\]

This problem is not convex in the standard form since \( f(x_1) = 1 - x_1 x_2 \) is not convex on \( \mathbb{R}^2 \).

But this problem can be cast as a standard convex problem:

\[
\begin{align*}
\text{min} & \quad x_1 + x_2 \\
\text{s.t.} & \quad -x_1 \leq 0 \\
 & \quad -x_2 \leq 0 \\
 & \quad 1 - \sqrt{x_1 x_2} \leq 1 \quad \text{(convex)} \\
 & \quad -\log x_1 - \log x_2 \leq 0.
\end{align*}
\]

\[ x^* = (1, 1) \]

\[ p^* = 2. \]

A convex optimization problem can also be written as

\[
\begin{align*}
\text{min} & \quad f(x) \\
\text{s.t.} & \quad f_i(x) \leq 0 \quad i = 1, \ldots, m \\
 & \quad Ax = b \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m.
\end{align*}
\]
1) Local and global optimality:
For a convex optimization problem, a fundamental property is that any locally optimal point is also globally optimal.

This property is easy to see:
By contradiction, suppose there exists a point y outside the local circle of radius R centered at x, such that f(y) < f(x).

Take a small step from x towards y: z = (1-\theta)x + \theta y such that z is still inside the circle.
Then f(z) \leq (1-\theta)f(x) + \theta f(y) < f(x)
contradictory to the assumption that x is locally optimal.

2) Optimality criteria for differentiable f:
Recall the first-order condition of convex functions:
\[ f(y) \geq f(x) + \nabla f(x)(y-x) \quad \forall x, y \in \text{dom } f. \]

Then for x to be optimal, it must satisfy
\[ \nabla f(x^*)(y-x^*) \geq 0, \quad \forall y \text{ feasible}. \]
The converse is also true (i.e., this is an iff statement).

This condition means that:
either \( \nabla f(x^*) = 0 \)
or \( -\nabla f(x^*) \) defines a supporting hyperplane to the feasible set at \( x^* \).
Special cases:

- **Unconstrained problems:** The optimality condition reduces to the well-known condition

  \[ x^* \text{ optimal } \iff \nabla f(x^*) = 0. \]

  This is because for unconstrained problems, the feasible set is the domain of \( f(x) \).

  For differentiable \( f(x) \), then its domain is an open set. Therefore, all points surrounding \( x^* \) and sufficiently close are feasible.

  Thus let \( y = x^* - t\nabla f(x^*) \), for \( t > 0 \) and small enough

  then \( y \in \text{dom}\ f \rightarrow \nabla f_0(x^*)^T(y-x^*) = -t\|\nabla f_0(x^*)\|^2 \)

  \( \rightarrow \nabla f_0(x^*) = 0 \).

**Example:**

\[
\min f_0(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \quad a_i \in \mathbb{R}^n.
\]

\( \text{dom} f_0 = \{ x \mid Ax < b \} \) is an open set.

Thus the optimality conditions are:

\[
Ax < b \quad \nabla f_0(x) = \sum_{i=1}^{m} \frac{1}{b_i - a_i^T x} a_i = 0.
\]

Then we solve this set of equations and inequalities to find \( x^* \).

- **Problems with equality constraints only:**

\[
\min f_0(x) \quad \text{s.t.} \quad Ax = b, \quad A \in \mathbb{R}^{p \times n}
\]

The optimality condition is

\[
\nabla f_0(x)^T(y - x) \geq 0 \quad \text{for all} \quad y : Ay = b.
\]

Since \( x, y \) are feasible \( \rightarrow y = x + w \) for \( w \in \text{N}(A) \), the null space of \( A \).

Then, the optimality condition can be written as
\[ \nabla f_0(x)^T w \geq 0 \quad \forall w \in \mathcal{N}(A) \]
\[ \Rightarrow \nabla f_0(x)^T w = 0 \quad \forall w \in \mathcal{N}(A) \quad (\text{since } \mathcal{N}(A) \text{ is a subspace, which contains } \omega) \]
Thus \( \nabla f_0(x) \perp \mathcal{N}(A) \) or \( \nabla f_0(x) \in \mathcal{N}(A)^\perp = \mathcal{R}(A^T) \).
That is, \( \nabla f_0(x) \) can be written as \( \nabla f_0(x) = A^T u \) for some \( u \in \mathcal{R}^p \).
In other words, \( x \) is optimal iff there exists a \( u \in \mathcal{R}^p \) such that
\[ \nabla f_0(x) + A^T u = 0 \]
\[ A x = b \]
This is the classical Lagrange multiplier (more in the duality topic).

**1) Equivalent convex problems:**

Two optimization problems are equivalent if from a solution of one, a solution of the other is readily found, and vice versa.

Next we describe some general transformations that yield equivalent problems.
The first two are general transformations. The others are transformation that also preserve convexity.

**i) Change of variable:**

Suppose \( \Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a one-to-one function with image covering the problem domain \( D \), let \( x = \Phi(z) \) and define
\[ f_i(z) = f_i(\Phi(z)) \quad i = 0, \ldots, m \]
\[ h_i(z) = h_i(\Phi(z)) \quad i = 1, \ldots, p \]