Homework 1: Linear algebra concepts
Due: February 20, 2017

1. Suppose that $A \in \mathbf{C}^{n\times n}$ (the set of complex-valued matrices of size $n \times n$) is nonsingular. This statement is equivalent to saying that A has no eigenvalues equal to 0. If $\lambda \in \sigma(A)$ (the set of eigenvalues of A), show that $\lambda^{-1} \in \sigma(A^{-1})$. If $Ax = \lambda x$ and $x \neq 0$, give an eigenvector of A^{-1} associated with λ^{-1}.

2. Let $A \in \mathbf{R}^{n\times n}$. If λ is a real eigenvalue of A with $Ax = \lambda x$, $0 \neq x \in \mathbf{C}^n$, let $x = \zeta + i\eta$, where $\zeta, \eta \in \mathbf{R}^n$ are the entrywise real and imaginary parts of x. Show that $A\zeta = \lambda\zeta$ and $A\eta = \lambda\eta$; conclude that there is a real eigenvector of A associated with λ. Must both ζ and η be eigenvectors of A? Can there be a real eigenvector associated with a complex non-real eigenvalue of A?

3. A matrix $A \in \mathbf{C}^{n\times n}$ is called Hermitian if $A^* = A$. If A is Hermitian, show that all eigenvalues of A are real. Hint: Let $\lambda \in \sigma(A)$ be arbitrary, and let x be an associated eigenvector. Then examnine the relation $x^*Ax = \lambda x^*x$.

4. For matrices $A \in \mathbf{C}^{m\times n}$ and $B \in \mathbf{C}^{n\times m}$, show by direct calculation that $\mathrm{tr}AB = \mathrm{tr}BA$. Use this fact to show that for $A \in \mathbf{C}^{n\times n}$ and nonsingular $S \in \mathbf{C}^{n\times n}$, $\mathrm{tr}S^{-1}AS = \mathrm{tr}A$. The matrix $S^{-1}AS$ is called a similarity of A, and this result says that the trace is a similarity invariant.

5. Let $A, B \in \mathbf{C}^{n\times n}$. If B is similar to A, show that the characteristic polynomial of B is the same as that of A. Show that rank is also a similarity invariant: If B is similar to A, then rank B = rank A.

6. If $A \in \mathbf{S}^n$ (the set of symmetric matrices of size $n \times n$). Show that

$$\mathrm{tr}A^k = \sum_{i=1}^{n} \lambda_i^k \tag{1}$$

for all positive integers k. The right-hand sum is called the k^{th} moment of the eigenvalues of A. (Note that this result also holds for a generic square matrix in $\mathbf{C}^{n\times n}$ but the proof is more involved.)

7. Let $A, B \in \mathbf{C}^{n\times n}$, A is non-singular. Find the vector x to maximize the following function:

$$f(x) = \frac{x^*Bx}{x^*Ax} \tag{2}$$

where x^* denotes the conjugate transpose of x. What is the maximum value of $f(x)$?