Each centering step will have the convergence analysis the same as of Newton's method (requires technical conditions).

- Can also be infeasible (start for each centering step, the center point $x^*(t)$ will be feasible.

- Numerical evidence suggests that each centering step takes a nearly constant number of Newton steps. So centering step doesn't become more difficult as $t$ increases, since the previous step gives a good starting point for the next one.

See figures 11.4, 11.5, 11.6 in the text for example of convergence rate from real problems. Also 11.7, 11.8.

Lecture 24:

- Feasibility and phase I methods

- The barrier method requires a strictly feasible starting point $x^0$.

- If we don't know this point, then we need to find (compute) it! This stage is called phase I which proceed the barrier method.

Phase I - compute a strictly feasible point or find that problem is infeasible.

Phase II - barrier method.

- Feasibility problem:

$$\min \quad \mathcal{S}$$

$$\mathcal{S} + f_i(x) \leq \mathcal{S} \quad i = 1 \ldots m$$

$$Ax = b$$
This feasibility problem is always feasible:
- Assume we are given \( x^{(0)} \) such that
  \[
  Ax^{(0)} = b
  \]
  and \( x^{(0)} \in \text{domf}_1 \cap \cdots \cap \text{domf}_n \).
- Then start with \( x^{(0)} \), take \( s > \max \{ f_i(x^{(0)}) \} \)
- we got a strictly feasible starting point.
Thus we can apply the barrier method on this problem.

- Three cases of optimal \( \bar{p}^* \) of the feasibility problem
  - (i) if \( \bar{p}^* < 0 \): then we have a strictly feasible point for the original problem.
  - (ii) if \( \bar{p}^* > 0 \): original problem is infeasible.
    - can construct a dual feasible point with positive dual objective to prove it.
  - (iii) if \( \bar{p}^* = 0 \): If \( \bar{p}^* \) is attained and \( s^* = 0 \) then
    - the set of inequality is feasible but in strictly feasible.
    - If \( \bar{p}^* \) is not attained \( \Rightarrow \) inequalities are infeasible.

In practice we cannot determine if \( \bar{p}^* = 0 \), but can only determine up to \( |p^*| < \epsilon \) for some small \( \epsilon > 0 \).

4) Some variation of phase I method:
\[
\text{min } \left\{ \sum s_i \right\}
\]
\[
\text{s.t. } \begin{align*}
  f_i(x) & \leq s_i, \\
  Ax & = b, \\
  s_i & \geq 0
\end{align*}
\]
Optimal value of \( s \) is 0 when the original system is feasible.
When the system is infeasible, sum of infeasibilities give the number of inequalities that are satisfied and the number of inequalities that are infeasible.

This indication of number of feasible/infeasible inequalities is usually better than using the max infeasibility method.

+ Phase I via infeasible start Newton method: Rewrite the original as:
  \[
  \min f_0(x) \\
  \text{s.t. } f_i(x) \leq 0 \quad i = 1, \ldots, m \\
  Ax = 0, \quad s = 0
  \]

  Then start the barrier method:
  \[
  \min \left[ f_0(x) - \sum_{i=1}^{m} \log(s - f_i(x)) \right] \\
  \text{s.t. } Ax = b, \quad s = 0
  \]

  and use the infeasible start Newton's method with any initial \( x \in D \) and \( s > \max_i f_i(x) \).

  Provided that the problem is strictly feasible, this infeasible start Newton's method will eventually take a full step and produce \( s = 0 \) and \( x \) strictly feasible.

+ Example of barrier/central path method:

  \[
  \min c^T x \\
  \text{s.t. } Ax \leq b
  \]

  The log barrier is:
  \[
  \phi(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x) \quad \text{dom } \phi = \{x | Ax = b\}.
  \]
+ Phase I: We need to solve the feasibility problem:

\[
\begin{align*}
\min & \quad s \\
\text{s.t.} & \quad Ax \leq b + s_i \\
& \quad \text{use barrier method}
\end{align*}
\]

or solve

\[
\min \quad - \sum_{i=1}^{m} \log s_i
\]

\[
\text{s.t.} \quad Ax + s = b
\]

using infeasible start Newton's method. If problem is feasible it will produce \( s > 0 \) and \( Ax < b \).

If the problem is on the boundary of feasibility and infeasibility, the computational complexity (# iterations grows fast). If the problem is exactly feasible but not strictly feasible, the computational complexity is np-hard.

+ Phase II: Assume that now we have identified a strictly feasible point \( x^{(0)} \).

Gradient and Hessian of \( \Phi \):

\[
\nabla \Phi(x) = \sum_{i=1}^{m} \frac{1}{b_i - a_i^T x} a_i \\
\nabla^2 \Phi(x) = \sum_{i=1}^{m} \frac{1}{(b_i - a_i^T x)^2} a_i a_i^T
\]

or as

\[
\nabla \Phi(x) = A^T d \\
\n\nabla^2 \Phi(x) = A^T \text{diag}(d) A
\]

where \( d_i = \frac{1}{b_i - a_i^T x} \), \( d > 0 \).

The centrality condition is:

\[
0 = t c + \nabla \Phi(x) = t c + A^T d = 0
\]

Geometric interpretation:

- \( \nabla \Phi(x) \) must be parallel to \(-c\) (since \( t > 0 \)).
- \( \nabla \Phi(x^*(t)) \) normal to level set of \( \Phi \) through \( x^*(t) \).

\[
\Rightarrow c^T x = c^T x^*(t) \quad \text{must be tangent to level set of } \Phi \text{ through } x^*(t) \]

(hyper-plane)