Homework 1

1. Here is a statement about pairwise independence and joint independence. Let X, Y_1 and Y_2 be binary random variables. If $I(X;Y_1) = 0$ and $I(X;Y_2) = 0$, does it follow that $I(X;Y_1,Y_2) = 0$?

   (a) Yes or no?

   (b) Prove or provide a counterexample.

   (c) If $I(X;Y_1) = 0$ and $I(X;Y_2) = 0$ in the above problem, does it follow that $I(Y_1;Y_2) = 0$? In other worlds, if Y_1 is independent of X, and of Y_2 is independent of X, is it true that Y_1 and Y_2 are independent?

2. Consider a sequence of n binary random variables $X_1, X_2, \cdots, X_n$. Each n-sequence with an even number of 1's has probability $2^{-(n-1)}$ and each n-sequence with an odd number of 1's has probability 0. Find the mutual informations

$$I(X_1;X_2),\ I(X_2;X_3|X_1),\ \cdots,\ I(X_{n-1};X_n|X_1,\cdots,X_{n-2})$$

3. Let X, Y and Z be joint random variables.

   (a) Prove the following inequality and find conditions for equality

$$I(X;Z|Y) \geq I(Z;Y|X) - I(Z;Y) + I(X;Z)$$

   (b) Give examples of X, Y and Z for the following inequalities

   - $I(X;Y|Z) < I(X;Y)$
   - $I(X;Y|Z) > I(X;Y)$

4. Csiszár's sum identity is given as follows.

$$\sum_{i=1}^{n} I(X_{i+1}^n;Y_i|Y^{i-1}) = \sum_{i=1}^{n} I(Y^{i-1};X_i|X_{i+1}^n)$$

   where $X_{n+1}, Y_0 = \emptyset$. Prove this identity.

5. An n-dimensional rectangular box with sides $X_1, X_2, \cdots, X_n$ is to be constructed. The volume is $V_n = \prod_{i=1}^n X_i$. The edge-length l of an n-cube with the same volume as the random box is $l = V_n^{1/n}$. Let $X_1, X_2, \cdots$ be i.i.d. uniform random variables over the interval $[0,a]$.

   Find $\lim_{n\to\infty} V_n^{1/n}$, and compare to $(EV_n)^{1/n}$. Clearly the expected edge length does not capture the idea of the volume of the box.

6. Suppose that (X, Y, Z) are jointly Gaussian and that $X \to Y \to Z$ forms a Markov chain. Let X and Y have correlation coefficient ρ_1 and let Y and Z have correlation coefficient ρ_2. Find $I(X; Z)$.

7. Let $Y = X_1 + X_2$. Find the maximum entropy (over all distributions on X_1 and X_2) of Y under the constraint $E[X_1^2] = P_1$, $E[X_2^2] = P_2$.
   (a) if X_1 and X_2 are independent.
   (b) if X_1 and X_2 are allowed to be dependent.