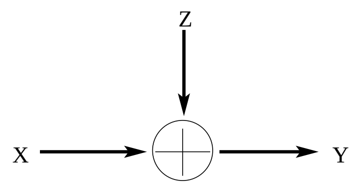
Homework 2

1. Find the channel capacity of the following discrete memoryless channel:

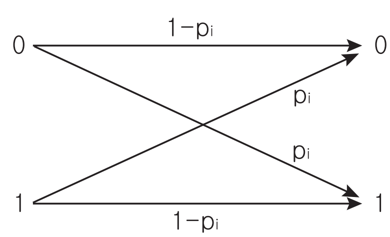


where $\Pr\{Z=0\} = \Pr\{Z=a\} = \frac{1}{2}$. The alphabet for x is $\mathbf{X} = \{0,1\}$. Assume that Z is independent of X. What is the optimal input distribution $p^*(x)$ that achieves the capacity? Observe that the channel capacity depends on the value of a.

2. Using two channels.

   Find the capacity C of the *union* 2 channels $(\mathcal{X}_1, p(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p(y_2|x_2), \mathcal{Y}_2)$ where, at each time, one can send a symbol over channel 1 or channel 2 but not both. Assume the output alphabets are distinct and do not intersect. Show $2^C = 2^{C_1} + 2^{C_2}$.

3. Consider a time-varying discrete *memoryless* binary symmetric channel. Let $Y_1, Y_2, \cdots, Y_n$ be conditionally independent given $X_1, X_2, \cdots, X_n$, with conditional distribution given by $p(y^n|x^n) = \prod_{i=1}^{n} p_i(y_i|x_i)$, as shown below.



   (a) Find $\max_{p(x)} I(X^n; Y^n)$.

   (b) We now ask for the capacity for the time invariant version of this problem. Replace each p_i, $1 \le i \le n$, by the average value $\bar{p} = \frac{1}{n}\sum_{j=1}^{n} p_j$, and compare the capacity to part (a).

4. Consider the ordinary additive noise Gaussian channel with two correlated looks at X, i.e., $Y = (Y_1, Y_2)$, where

$$Y_1 = X + Z_1$$
$$Y_2 = X + Z_2$$

   with a power constraint P on X, and $(Z_1, Z_2) \sim \mathcal{N}_2(\mathbf{0}, K)$, where

$$K = \begin{bmatrix} N & \rho N \\ \rho N & N \end{bmatrix}$$

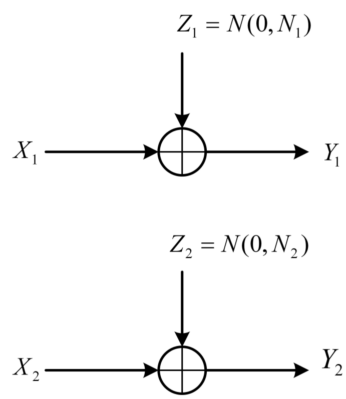
   Find the capacity C for

(a) $\rho = 1$

(b) $\rho = 0$

(c) $\rho = -1$

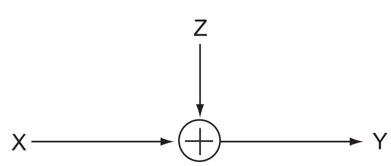
5. Consider the following parallel Gaussian channel in the figure below where $Z_1 \sim \mathcal{N}(0, N_1)$, $Z_2 \sim \mathcal{N}(0, N_2)$, and Z_1 and Z_2 are independent Gaussian random variables and $Y_i = X_i + Z_i$. We wish to allocate power to the two parallel channels. Let β_1 and β_2 be fixed. Consider a total cost constraint $\beta_1 P_1 + \beta_2 P_2 \leq \beta$, where P_i is the power allocated to the i_{th} channel and β_i is the cost per unit power in that channel. Thus, $P_1 \geq 0$ and $P_2 \geq 0$ can be chosen subject to the cost constraint β.



(a) For what value of β does the channel stop acting like a single channel and start acting like a pair of channels?

(b) Evaluate the capacity and find P_1 and P_2 that achieve capacity for $\beta_1 = 1, \beta_2 = 2, N_1 = 3, N_2 = 2$, and $\beta = 10$.

6. Consider the following channel:



Throughout this problem we shall constrain the signal power

$$E[X] = 0, \quad E[X^2] = P,$$

and the noise power

$$E[Z] = 0, \quad E[Z^2] = N,$$

and assume that X and Z are independent. The channel capacity is given by $I(X; X + Z)$.

Now for the game. The noise player chooses a distribution on Z to minimize $I(X; X + Z)$, while the signal player chooses a distribution on X to maximize $I(X; X + Z)$. Letting $X^* \sim \mathcal{N}(0, P)$, $Z^* \sim \mathcal{N}(0, N)$, show that Gaussian X^* and Z^* satisfy the saddle point conditions

$$I(X; X + Z^*) \leq I(X^*; X^* + Z^*) \leq I(X^*; X^* + Z).$$

Thus

$$\min_Z \max_X I(X; X+Z) = \max_X \min_Z I(X; X+Z) = \frac{1}{2}\log\left(1+\frac{P}{N}\right),$$

and the game has a value. In particular, a deviation from normal for either player worsens the mutual information from that player's standpoint. Can you discuss the implications of this?

Note: Part of the proof hinges on the entropy power inequality from Chapter 16, which states that if $\mathbf{X}$ and $\mathbf{Y}$ are independent random n-vectors with densities, then

$$e^{\frac{2}{n}h(\mathbf{X}+\mathbf{Y})} \geq e^{\frac{2}{n}h(\mathbf{X})} + e^{\frac{2}{n}h(\mathbf{Y})}$$

7. A train pulls out of the station at constant velocity. The received signal energy thus falls off with time as $1/i^2$. The total received signal at time i is

$$Y_i = \frac{1}{i}X_i + Z_i$$

where $Z_1, Z_2, \cdots$ are i.i.d. $\sim \mathcal{N}(0, N)$. The transmitter constraint for block length n is

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2(w) \leq P, \quad w \in \{1, 2, \cdots, 2^{nR}\}$$

Using Fano's inequality, show that the capacity C is equal to zero for this channel.

8. Consider the vector Gaussian noise channel

$$Y = X + Z$$

where $X = (X_1, X_2, X_3)$, $Z = (Z_1, Z_2, Z_3)$, $Y = (Y_1, Y_2, Y_3)$, $E[||X||^2] \leq P$, and

$$Z \sim \mathcal{N}\left(0, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}\right)$$

Find the capacity. The answer may be surprising.

9. Joint typicality theorem and Packing lemma.

   (a) Let (X_i, Y_i) be i.i.d. according to $p(x, y)$. We say that (x^n, y^n) is jointly typical (written $(x^n, y^n) \in A_\epsilon^{(n)}$) if all the following inequalities hold:

$$\begin{aligned} 2^{-n(H(X)+\epsilon)} &\leq p(x^n) \leq 2^{-n(H(X)-\epsilon)} \\ 2^{-n(H(Y)+\epsilon)} &\leq p(y^n) \leq 2^{-n(H(Y)-\epsilon)} \\ 2^{-n(H(X,Y)+\epsilon)} &\leq p(x^n, y^n) \leq 2^{-n(H(X,Y)-\epsilon)} \end{aligned}$$

   Now suppose that $(\tilde{X}^n, \tilde{Y}^n)$ is drawn according to $p(x^n)p(y^n)$. Thus, $(\tilde{X}^n, \tilde{Y}^n)$ have the same marginals as (X^n, Y^n) (which were drawn according to $p(x^n, y^n)$) but are independent. Prove that

$$\Pr\{(\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon^{(n)}\} \leq 2^{-n(I(X;Y)-3\epsilon)}.$$

(b) Let (U, X, Y) be i.i.d. according to $p(u, x, y)$. Let $(\tilde{U}^n, \tilde{Y}^n) \sim p(\tilde{u}^n, \tilde{y}^n)$ be a pair of arbitrary distributed random sequences (not necessarily according to $\prod_{i=1}^n p_{U,Y}(\tilde{u}_i, \tilde{y}_i)$). Let $X^n(m), m \in [1, 2^{nR}]$ be random sequences each distributed according to $\prod_{i=1}^n p_{X|U}(x_i|\tilde{u}_i)$. Assume that $X^n(m)$ is pairwise conditionally independent of $\tilde{Y}^n$ given $\tilde{U}^n$, but is arbitrarily dependent on other $X^n(m)$ sequences.

Prove that

$$\Pr\{(\tilde{U}^n, X^n(m), \tilde{Y}^n \in A_\epsilon^{(n)}\} \to 0$$

for some $m \in [1, 2^{nR}]$ as $n \to \infty$ if

$$R < I(X; Y|U) - \epsilon.$$