Homework 3 – The Multiple Access Channel

1. Capacity region of simple multiple access channels.
   (a) What is the capacity of the following multiple access channel?

   \[ X_1 \in \{-1, 0, 1\} \]
   \[ X_2 \in \{-1, 0, 1\} \]
   \[ Y = X_1^2 + X_2^2 \]

   i. Find the capacity region.
   ii. Describe \( p^*(x_1), p^*(x_2) \) achieving a point on the boundary of the capacity region.
   iii. What is the capacity if \( Y = X_1X_2 \)?

   (b) Find the capacity region of the modulo-2 sum MAC, where \( X_1 \) and \( X_2 \) are binary and \( Y = X_1 \oplus X_2 \). Show that the capacity region can be expressed as the union of \( \mathcal{R}(X_1, X_2) \) sets and therefore time sharing is not necessary.

2. MAC capacity as cutset bounds. For the multiple access channel we know that \( (R_1, R_2) \) is achievable if

   \[ R_1 < I(X_1; Y | X_2) \]
   \[ R_2 < I(X_2; Y | X_1) \]
   \[ R_1 + R_2 < I(X_1, X_2; Y) \]

   for \( X_1, X_2 \) independent. Show, for \( X_1, X_2 \) independent, that

   \[ I(X_1; Y | X_2) = I(X_1; Y, X_2) \]

   Thus \( R_1 \) is less than the mutual information between \( X_1 \) and everything else.

   Interpret the information bounds as bounds on the rate of flow across cutsets \( S_1, S_2 \) and \( S_3 \).

3. Achievable SNR region. Consider a Gaussian multiple access channel \( Y = gX_1 + X_2 + Z \) with \( Z \sim N(0, 1), g \geq 1 \), and the power constraints \( P_1 \) on \( X_1 \) and \( P_2 \) on \( X_2 \).

   (a) Specify the capacity region of this channel with SNRs \( S_1 = g^2P_1 \) and \( S_2 = P_2 \).
(b) Suppose we wish to communicate reliably at a fixed rate pair \((R_1, R_2)\). Specify the achievable SNR region \(S(R_1, R_2)\) consisting of all SNR pairs \((s_1, s_2)\) such that \((R_1, R_2)\) is achievable.

(c) Find the SNR pair \((s_1^*, s_2^*)\) \(\in S(R_1, R_2)\) that minimizes the total average transmission power \(P_{\text{sum}} = P_1 + P_2\). (Hint: You can use a simple geometric argument.)

(d) Can \((R_1, R_2)\) be achieved with minimum total average power \(P_{\text{sum}}\) using only successive cancellation decoding (i.e., without time sharing)? If so, what is the order of message decoding? Can \((R_1, R_2)\) be achieved by time division with power control? (Hint: Treat the cases \(g = 1\) and \(g > 1\) separately.)

(e) Find the minimum-energy-per-bit region of the Gaussian MAC, that is, the set of all energy pairs \((E_1, E_2) = (P_1/R_1, P_2/R_2)\) such that the rate pair \((R_1, R_2)\) is achievable with average code power pair \((P_1, P_2)\).

4. **Multiple-antenna Gaussian MAC.** Consider a two-user Gaussian multiple access channel with channel output \(Y = (Y_1, Y_2)\) given by

\[
Y_1 = X_1 + Z_1 \\
Y_2 = X_1 + X_2 + Z_2,
\]

where channel inputs \(X_1\) and \(X_2\) from each user are both subject to power constraint \(P\), and the zero-mean unit-variance Gaussian noises \(Z_1\) and \(Z_2\) are independent of each other and channel inputs.

(a) Find the capacity region.

(b) Find the time-division region with power control. Is it possible to achieve any point on the boundary of the capacity region (except for the end points) using time-division?

5. **Capacity of Gaussian MAC with rate splitting.** The capacity of a multiple access channel can be achieved by successive cancellation decoding to achieve the corner points of the capacity region, and time-sharing to achieve the rate points in between. Instead of time-sharing, another technique that can achieve the capacity of the Gaussian MAC is **rate splitting** at one of the encoder, and then use successive single-user decoding. The idea is to split one encoder (say encoder 2) into two encoders operating at the respective rates \(R_{21}\) and \(R_{22}\), where \(R_2 = R_{21} + R_{22}\). Suppose that these encoders generate codewords with respective powers \(P_{21}\) and \(P_{22}\), where \(P_2 = P_{21} + P_{22}\). Then the transmitted codeword from user 2 is the sum of the two codewords generated by these two encoders. The decoder performs single-user decoding in three stages: first, decode the \(R_{21}\) code; second, decode the \(R_1\) code; third, decode the \(R_{22}\) code.

(a) Establish the achievable rate region for the Gaussian MAC using this rate-splitting and single-user decoding technique.

(b) Show that this rate region equals to the known MAC capacity region.

Notes: This rate-splitting approach to the Gaussian MAC was established in Bixio Rimoldi and Rudiger Urbanke, "A Rate-Splitting Approach to the Gaussian Multiple-Access Channel" IEEE Trans. on Info. Theory, Vol. 46, NO. 2, 1996.

6. **Multiple access channel with degraded message sets.** Consider a DM-MAC \((X_1 X_2, p(y|x_1, x_2), Y)\). Sender 1 encodes two independent messages \((M_0, M_1)\) uniformly distributed over \([2^{nR_0}] \times [2^{nR_1}]\), while Sender 2 encodes the message \(M_0\) only. Thus, the common message \(M_0\) is available to both senders, while the private message \(M_1\) is available only to Sender 1. The receiver \(Y\) needs to decode both \(M_0\) and \(M_1\).
The capacity region of this channel is given by the convex closure of all rate pairs \((R_1, R_2)\) satisfying

\[
R_1 < I(X_1; Y | X_2) \\
R_1 + R_0 < I(X_1, X_2; Y)
\]

for some \(p(x_1, x_2)p(y|x_1, x_2)\).

(a) Prove the achievability of the capacity region.
(b) Prove the weak converse of the capacity region.
(c) What is the capacity region for the AWGN-MAC with degraded message sets under noise power 1 and input power constraints \(P_1\) and \(P_2\)?