Homework 3 – The Multiple Access Channel

1. **Capacity region of simple multiple access channels.**

   (a) What is the capacity of the following multiple access channel?

   $$X_1 \in \{-1, 0, 1\}$$
   $$X_2 \in \{-1, 0, 1\}$$
   $$Y = X_1^2 + X_2^2$$

      i. Find the capacity region.

      ii. Describe $p^*(x_1)$, $p^*(x_2)$ achieving a point on the boundary of the capacity region.

      iii. What is the capacity if $Y = X_1 X_2$?

   (b) Find the capacity region of the modulo-2 sum MAC, where X_1 and X_2 are binary and $Y = X_1 \oplus X_2$. Show that the capacity region can be expressed as the union of $\mathcal{R}(X_1, X_2)$ sets and therefore time sharing is not necessary.

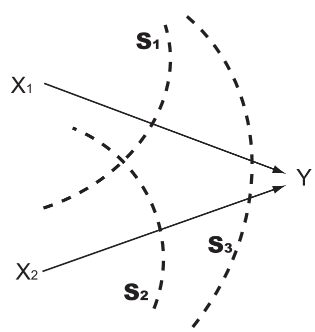
2. **MAC capacity as cutset bounds.** For the multiple access channel we know that (R_1, R_2) is achievable if

   $$R_1 < I(X_1; Y|X_2)$$
   $$R_2 < I(X_2; Y|X_1)$$
   $$R_1 + R_2 < I(X_1, X_2; Y)$$

   for X_1, X_2 independent. Show, for X_1, X_2 independent, that

   $$I(X_1; Y|X_2) = I(X_1; Y, X_2)$$

   Thus R_1 is less than the mutual information between X_1 and everything else.



   Interpret the information bounds as bounds on the rate of flow across cutsets S_1, S_2 and S_3.

3. **Achievable SNR region.** Consider a Gaussian multiple access channel $Y = gX_1 + X_2 + Z$ with $Z \sim N(0, 1)$, $g \geq 1$, and the power constraints P_1 on X_1 and P_2 on X_2.

   (a) Specify the capacity region of this channel with SNRs $S_1 = g^2 P_1$ and $S_2 = P_2$.

(b) Suppose we wish to communicate reliably at a fixed rate pair (R_1, R_2). Specify the *achievable SNR region* $\mathcal{S}(R_1, R_2)$ consisting of all SNR pairs (s_1, s_2) such that (R_1, R_2) is achievable.

(c) Find the SNR pair $(s_1^*, s_2^*) \in \mathcal{S}(R_1, R_2)$ that minimizes the total average transmission power $P_{\text{sum}} = P_1 + P_2$. (Hint: You can use a simple geometric argument.)

(d) Can (R_1, R_2) be achieved with minimum total average power P_{sum} using only successive cancellation decoding (i.e., without time sharing)? If so, what is the order of message decoding? Can (R_1, R_2) be achieved by time division with power control? (Hint: Treat the cases $g = 1$ and $g > 1$ separately.)

(e) Find the minimum-energy-per-bit region of the Gaussian MAC, that is, the set of all energy pairs $(E_1, E_2) = (P_1/R_1, P_2/R_2)$ such that the rate pair (R_1, R_2) is achievable with average code power pair (P_1, P_2).

4. **Multiple-antenna Gaussian MAC.** Consider a two-user Gaussian multiple access channel with channel output $Y = (Y_1, Y_2)$ given by

$$\begin{aligned} Y_1 &= X_1 + Z_1 \\ Y_2 &= X_1 + X_2 + Z_2, \end{aligned}$$

where channel inputs X_1 and X_2 from each user are both subject to power constraint P, and the zero-mean unit-variance Gaussian noises Z_1 and Z_2 are independent of each other and channel inputs.

(a) Find the capacity region.

(b) Find the time-division region with power control. Is it possible to achieve any point on the boundary of the capacity region (except for the end points) using time-division?

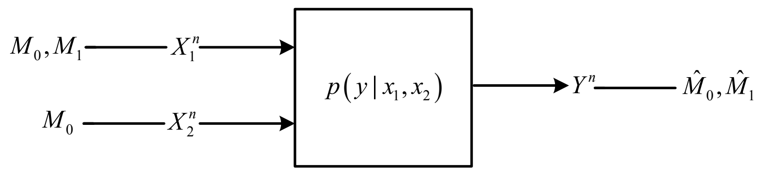
5. **Capacity of Gaussian MAC with rate splitting.** The capacity of a multiple access channel can be achieved by successive cancellation decoding to achieve the corner points of the capacity region, and time-sharing to achieve the rate points in between. Instead of time-sharing, another technique that can achieve the capacity of the Gaussian MAC is *rate splitting* at one of the encoder, and then use successive single-user decoding. The idea is to split one encoder (say encoder 2) into two encoders operating at the respective rates R_{21} and R_{22}, where $R_2 = R_{21} + R_{22}$. Suppose that these encoders generate codewords with respective powers P_{21} and P_{22}, where $P_2 = P_{22} + P_{22}$. Then the transmitted codeword from user 2 is the sum of the two codewords generated by these two encoders. The decoder performs single-user decoding in three stages: first, decode the R_{21} code; second, decode the R_1 code; third, decode the R_{22} code.

(a) Establish the achievable rate region for the Gaussian MAC using this rate-splitting and single-user decoding technique.

(b) Show that this rate region equals to the known MAC capacity region.

Notes: This rate-splitting approach to the Gaussian MAC was established in Bixio Rimoldi and Rudiger Urbanke, "A Rate-Splitting Approach to the Gaussian Multiple-Access Channel" IEEE Trans. on Info. Theory, Vol. 46, NO. 2, 1996.

6. **Multiple access channel with degraded message sets.** Consider a DM-MAC $(\mathcal{X}_1\mathcal{X}_2, p(y|x_1, x_2), \mathcal{Y})$. Sender 1 encodes two independent messages (M_0, M_1) uniformly distributed over $[2^{nR_0}] \times [2^{nR_1}]$, while Sender 2 encodes the message M_0 only. Thus, the common message M_0 is available to both senders, while the private message M_1 is available only to Sender 1. The receiver Y needs to decode both M_0 and M_1.



Multiple access channel with DMS.

The capacity region of this channel is given by the convex closure of all rate pairs (R_1, R_2) satisfying

$$R_1 < I(X_1; Y|X_2)$$
$$R_1 + R_0 < I(X_1, X_2; Y)$$

for some $p(x_1, x_2)p(y|x_1, x_2)$.

(a) Prove the achievability of the capacity region.

(b) Prove the weak converse of the capacity region.

(c) What is the capacity region for the AWGN-MAC with degraded message sets under noise power 1 and input power constraints P_1 and P_2?