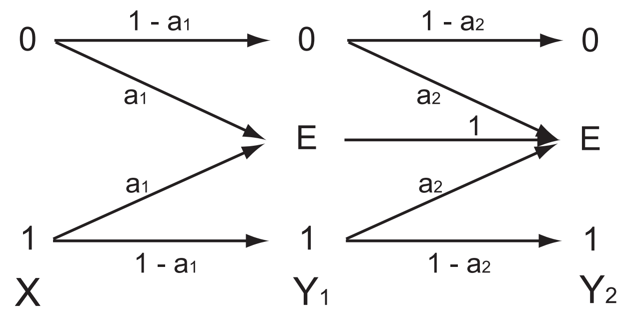
Homework 4 – The Broadcast Channel

1. **Degraded broadcast channel.** Consider the following degraded broadcast channel.



   (a) What is the capacity of the channel from X to Y_1?
   (b) What is the channel capacity from X to Y_2?
   (c) What is the capacity region of all (R_1, R_2) achievable for this broadcast channel? Simplify and sketch.

2. **Joint and independent receivers.** Assume that a sender X is sending to two fixed base stations. Assume that the sender sends a signal X that is constrained to have average power P. Assume that the two base stations receive signals Y_1 and Y_2, where

$$Y_1 = \alpha_1 X + Z_1$$
$$Y_2 = \alpha_2 X + Z_2$$

   where $Z_1 \sim \mathcal{N}(0, N_1)$, $Z_2 \sim \mathcal{N}(0, N_2)$, and Z_1 and Z_2 are independent. We will assume the α's are constant over a transmitted block.

   (a) Assuming that both signals Y_1 and Y_2 are available at a common decoder $Y = (Y_1, Y_2)$, what is the capacity of the channel from the sender to the common receiver?
   (b) If, instead, the two receivers Y_1 and Y_2 each decode their signals independently, this becomes a broadcast channel. Let R_1 be the rate to base station 1 and R_2 be the rate to base station 2. Find the capacity region of this channel.

3. **BC capacity points and practical superposition coding.**

   (a) For the degraded broadcast channel $X \to Y_1 \to Y_2$, find the points a and b where the capacity region hits the R_1 and R_2 axes. Show that $b \leq a$
   (b) Show a superposition coding of a 16-QAM constellation on top of a QPSK constellation.

4. **Duality between Gaussian BC and MAC.** Consider the following Gaussian broadcast and multiple access channels:

   The broadcast channel: At time i

$$Y_{1i} = g_1 X_i + Z_{1i}$$
$$Y_{2i} = g_2 X_i + Z_{2i},$$

where the Z_{1i} and Z_{2i} are *i.i.d.* and independent of X_i, $Z_{ki} \sim \mathcal{N}(0,1)$ for $k = 1, 2$. Assume average power constraint P on the codeword X^n.

The multiple access channel: At time i

$$Y_i = g_1 X_{1i} + g_2 X_{2i} + Z_i,$$

where the Z_i are *i.i.d.* and independent of X_i, $Z_i \sim \mathcal{N}(0,1)$. Assume a sum average power constraint on each pair of codewords $(x^n(m1), x^n(m2))$ as

$$\frac{1}{n}\sum_{i=1}^{n} x_{1i}^2 + x_{2i}^2 \leq P$$

(a) Provide expressions for the independent message capacity regions of these two channels in terms of $C(x) = \frac{1}{2}\log(1+x)$.

(b) Show that the two capacity regions are equal.

(c) Consider a point (R_1, R_2) on the boundary of the capacity region for the broadcast channel. Show that the same point exists on the boundary of the capacity for the multiple access channel with some power allocation between the two transmitters.

Using superposition coding for the broadcast channel that achieves a point on the boundary of its capacity region, argue that the same sequences of code can be used to achieve the same point on the capacity region of the given multiple access channel. What is special about the encoding order at the broadcast channel and the decoding order at the multiple access channel, assuming successive cancellation?

Notes: The above duality result is a simple case of a general duality result between Gaussian broadcast and multiple access channels established in N. Jindal, S. Vishwanath, A. J. Goldsmith, "On the duality of Gaussian multiple-access and broadcast channels," IEEE Trans. Inf. Theory, May 2004.

5. **Converse proof for the Gaussian BC.** Consider a Gaussian broadcast channel with $N_2 > N_1$. In the proof of the converse of the capacity theorem, Fano's inequality can be used to show that

$$nR_1 \leq I(M_1; Y_1^n | M_2) + n\epsilon_n$$
$$nR_2 \leq I(M_2; Y_2^n) + n\epsilon_n.$$

Now show that there exists an $\alpha \in [0,1]$ such that

$$I(M_2; Y_2^n) \leq nC\left(\frac{\bar{\alpha}P}{\alpha P + N_2}\right)$$
$$I(M_1; Y_1^n | M_2) \leq nC\left(\frac{\alpha P}{N_1}\right)$$

Hint: To prove the second inequality, first show that

$$I(M_1; Y_1^n | M_2) \leq h(Y_1^n | M_2) - \frac{n}{2}\log(2\pi e N_1),$$

then using a conditional version of the vector EPI given below, find an upper bound for $h(Y_1^n | M_2)$.

$$e^{\frac{2}{n}h((X^n+Y^n)|U)} \geq e^{\frac{2}{n}h(X^n|U))} + e^{\frac{2}{n}h(Y^n|U)}.$$

6. **The Blackwell broadcast channel.** The Blackwell broadcast channel is defined as follows.

$$\mathcal{X} = \{0,1,2\}, \mathcal{Y}_1 = \mathcal{Y}_2 = \{0,1\},$$
$$p(0,0|0) = p(0,1|1) = p(1,1|2) = 1.$$

   (a) Is this channel degraded? Why or why not?
   (b) Propose an achievable rate region for private messages using:
      i. Superposition coding
      ii. Marton's inner bound (over binning)
   (c) Find the Sato outer bound for this channel.
   (d) Plot all three bounds on a graph.

7. **More capable broadcast channel.** Let $(\mathcal{X}, p(y_1, y_2|x), \mathcal{Y}_1, \mathcal{Y}_2)$ be a DM-BC. Receiver Y_1 is said to be *more capable* than receiver Y_2 if $I(X;Y_1) \geq I(X;Y_2)$ for all input pmf $p(x)$. For the more capable broadcast channel, the capacity region is known and is achieved by superposition coding.

   (a) Show that if a broadcast channel is degraded then it is more capable, but not the other way around. Hence the more capable broadcast channel is a broader class than the degraded broadcast channel.
   (b) Show that if Y_1 is more capable than Y_2, then

$$I(X_n; Y_1^n) \geq I(X_n; Y_2^n)$$

   for all $p(x^n)$ and for all $n \geq 1$. (Hint: Assume without loss of generality that Y_1 and Y_2 are conditionally independent given X, and use induction.)
   (c) Specify the private-message rate region achievable by superposition coding for the more capable broadcast channel. Prove that it is also the capacity region.
   (d) Using part (a) to show that the independent message capacity region of the more capable broadcast channel is the set of all rate pairs (R_1, R_2) such that

$$R_2 \leq I(U;Y_2),$$
$$R_2 + R_1 \leq \min\{I(X;Y_1), I(X;Y_1|U) + I(U;Y_2)\}$$

   for some $p(u,x)$. What is the cardinality bound on U?