Homework 5

1. **Two-way channel.** The 2-way channel is a channel very similar to the interference channel, with the additional provision that sender 1 is attached to receiver 2 and sender 2 is attached to receiver 1, as shown in Figure 1. Hence, sender 1 can use information from previous received symbols of receiver 2 to decide what to send next.

   Consider the 2-way channel shown in Figure 1. Assume here that the outputs Y_1 and Y_2 depend *only* on the current inputs X_1 and X_2.

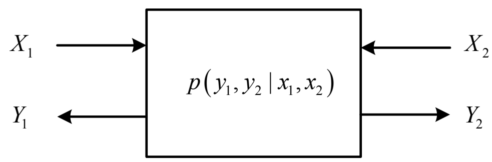


Figure 1: Two-way channel.

   (a) By using independently generated codes for the two senders, show that the following rate region is achievable:

$$R_1 < I(X_1; Y_2 | X_2),$$
$$R_2 < I(X_2; Y_1 | X_1),$$

   for some product distribution $p(x_1)p(x_2)p(y_1, y_2|x_1, x_2)$.

   (b) Show that the rates for any code for a two-way channel with arbitrarily small probability of error must satisfy

$$R_1 \le I(X_1; Y_2 | X_2),$$
$$R_2 \le I(X_2; Y_1 | X_1),$$

   for some joint distribution $p(x_1, x_2)p(y_1, y_2|x_1, x_2)$.

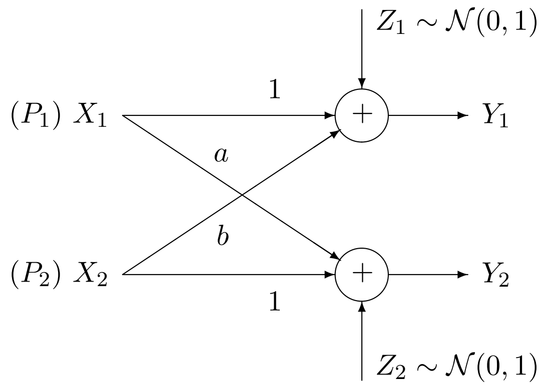
   The inner and outer bounds on the capacity of the two-way channel are due to Shannon. He also showed that the inner bound and the outer bound do not coincide in the case of the binary multiplying channel $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0, 1\}, Y_1 = Y_2 = X_1 X_2$. The capacity of the two-way channel is still an open problem.

2. **AWGN-IC.** Consider the following AWGN interference channel (AWGN-IC) model. At time i

$$Y'_{1i} = g_{11} X'_{1i} + g_{21} X'_{2i} + Z'_{1i}$$
$$Y'_{2i} = g_{12} X'_{1i} + g_{22} X'_{2i} + Z'_{2i},$$

   where Z'_{1i} and Z'_{2i} are discrete-time white Gaussian noise processes with average power N_1 and N_2 respectively, independent of X'_{1i} and X'_{2i}, and $g_{jk}, j, k = 1, 2$, are the channel gains. Let the codewords X_1^n and X_2^n have average power constraints of P'_1 and P'_2, respectively.

   (a) Convert the above channel to the standard AWGN-IC shown below with direct link gains of 1 and unit noise variances. What are the new codeword power constraints P_1 and P_2, and the new channel gains a and b? Provide arguments to why these two channels have the same capacity.



(b) For $a < 1$ and $b < 1$, derive the achievable rate regions for the standard AWGN-IC with the following coding schemes:

i. Time-sharing with power control

ii. Treating interference as Gaussian noise

iii. Simultaneous decoding of both messages at both receiver

iv. Han-Kobayashi rate splitting

Compare these rate regions. Does any of these regions include other regions? Prove or provide numerical examples (show a plot).

3. **Binary interference channel.** Consider the following binary interference channel:

$$Y_1 = (X_1 \oplus Z_1) \cdot X_2$$
$$Y_2 = (X_2 \oplus Z_2) \cdot X_1$$

where the inputs X_1 and X_2 are binary and independent of each other, the noises Z_1 and Z_2 are i.i.d. with distribution $\mathrm{Bern}(\alpha)$ and are independent of the inputs.

(a) Find the maximum rate of user 1.

(b) Evaluate the Han-Kobayashi region for this channel, using binary auxiliary random variables. (Note that binary auxiliary random variables may not be optimal, but are assumed here for simple computation.) Specify the encoding and decoding techniques and the optimal distributions of the auxiliary random variables that are used to achieve the boundary of this region.

4. **Equivalent inner bounds for the DM-IC.** For a discrete memoryless interference channel (DM-IC), the Han-Kobayashi (HK) achievable rate region is the convex closure of the rate tuple satisfying

$$\begin{aligned}
R_{11} &\le I(W_1; Y_1 | U_1, U_2, Q)\\
R_{10} &\le I(U_1; Y_1 | W_1, U_2, Q)\\
R_{20} &\le I(U_2; Y_1 | U_1, W_1, Q)\\
R_{11} + R_{10} &\le I(U_1, W_1; Y_1 | U_2, Q)\\
R_{11} + R_{20} &\le I(U_2, W_1; Y_1 | U_1, Q)\\
R_{10} + R_{20} &\le I(U_1, U_2; Y_1 | W_1, Q)\\
R_{11} + R_{10} + R_{20} &\le I(U_1, W_1, U_2; Y_1 | Q)
\end{aligned}$$

and 7 similar inequalities for R_{22}, R_{20}, R_{10} for some $(p(q)p(u_1|q)p(w_1|q)p(u_2|q)p(w_2|q)p(x_1|u_1, w_1, q)$ $p(x_2|u_2, w_2, q))$. (U_1, W_1, U_2, W_2) are auxiliary random variables serve to carry the messages $(M_{10}, M_{11}$, $M_{20}, M_{22})$, respectively.

On the other hand, the Chong-Motani-Garg (CMG) achievable rate region is the convex closure of the rate tuple satisfying

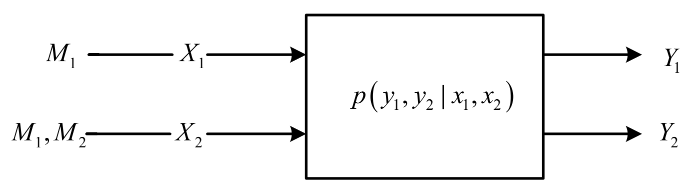
$$\begin{aligned} R_{11} &\leq I(X_1; Y_1|U_1, U_2, Q) \\ R_{11} + R_{10} &\leq I(X_1; Y_1|U_2, Q) \\ R_{11} + R_{20} &\leq I(U_2, X_1; Y_1|U_1, Q) \\ R_{11} + R_{10} + R_{20} &\leq I(X_1, U_2; Y_1|Q) \end{aligned}$$

and 4 similar inequalities for R_{22}, R_{20}, R_{10} for some $(p(q)p(u_1, x_1|q)\ p(u_2, x_2|q))$.

Simplify these rate constraints using the Fourier-Mozkin elimination process to obtain a rate region in terms of R_1 and R_2.

Show the equivalence between these two representations of the achievable rate regions.

5. **Cognitive interference channel** (also called IC with degraded message sets (IC-DMS)) The interference channel with degraded message set (IC-DMS) is an interference channel in which the message and codeword of one encoder is known non-causally at the other encoder. In other words, transmitter 2 encodes two independent messages (M_1, M_2) uniformly distributed over $[2^{nR_1}] \times [2^{nR_2}]$, while transmitter 1 encodes the message M_1 only. Receiver 1 needs to decode only M_1 and receiver 2 needs to decode only M_2. This channel is also called the *cognitive channel*.



Interference channel with DMS.

   (a) Use Gel'fand-Pinsker and Han-Kobayashi coding techniques, propose an achievable rate region for this channel.
   (b) What is this achievable rate region for the standard Gaussian IC with degraded message sets?

6. **Capacity gain from channel state information.** Consider the DMC with DM state $p(y|x, s)p(s)$. This problem compares between value of state information at the decoder and at the encoder.

   First, show that the capacity gain due to state information at the decoder is bounded by proving the following statements:

   (a) $C_{\text{SI-D}} - C_{\text{SI-none}} \leq \max_{p(x)} H(S|Y)$.
   (b) $C_{\text{SI-ED}} - C_{\text{SI-E}} \leq \max_{p(x|s)} H(S|Y)$.

   For part (b), assume the same type of encoder state information in both capacity expressions (both causal or both noncausal) and prove for both cases. These results show that the state information at the decoder is worth at most $H(S)$ bits.

   Next, show that the state information at the encoder can be much more valuable by providing an example where $C_{\text{SI-E}} - C_{\text{SI-none}} > H(S)$. (You can provide an example with either causal or non-causal state information.)

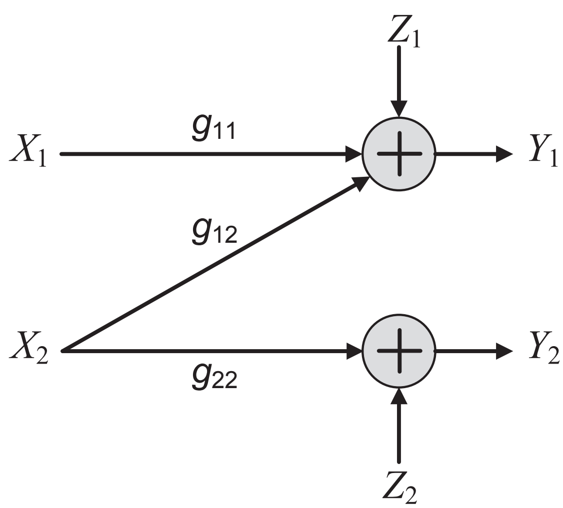
7. **Input optimization** Consider a binary discrete memoryless channel with discrete memoryless binary state S. When $S = 1$, the channel is a binary symmetric channel (BSC) with cross over probability α. When $S = 2$, the channel is a BSC with cross over probability β. Assume $\alpha < \beta < 0.5$. Let S be Bern(p), that is, $\Pr(S = 1) = p$ where $0 \leq p \leq 1$.

   Compute the capacity of this channel when the state is known non-causally only at the encoder. What is the capacity-achieving input distribution?

   For $\alpha = 0.01$ and $\beta = 0.89$, plot this capacity with as a function of p. Also plot and compare with the capacity with state known only at the decoder.

8. **Handoff** Consider two symmetric Gaussian ICs, one with SNR S and INR $I > S$, and the other with SNR I and INR S. Thus, the second Gaussian IC is equivalent to the setting where the messages are sent to the other receivers in the first Gaussian IC. Which has a larger capacity region?

9. **Gaussian Z interference channel** Consider the Gaussian IC depicted in the Figure below with SNRs S_1, S_2, and INR I_1. (Here the INR $I_2 = 0$.)



Gaussian interference channel with $I_2 = 0$.

   (a) Find the capacity region when $S_2 <= I_1$.
   (b) Find the sum-capacity when $I_1 <= S_2$.
   (c) Find the capacity region when $S_2 <= I_1/(1 + S_1)$.