Lecture 3: Gaussian Channels

1 Scalar Gaussian channel

Consider a discrete-time channel in which the input $X_i$ and output $Y_i$ are samples at time $i$ and are continuous signals. The noise samples $Z_i$ are i.i.d Gaussian, and $X_i$ and $Z_i$ are independent. At each sample, the channel can be expressed as

$$Y_i = X_i + Z_i$$

Without any constraint, the capacity is infinity.

Average power constraint

For each codeword $X^n = (X_1, X_2, \ldots, X_n)$

$$\frac{1}{n} \sum_{i=1}^{n} X_i^2 \leq P.$$  

We can also write $E[X^2] \leq P$ although this is slightly different as it implies the average power per column of the codebook (at each time sample).

Then using the result from the DMC

$$I(X; Y) = h(Y) - h(Y|X) = h(Y) - h(X + Z|X)$$
$$= h(Y) - h(Z|X) = h(Y) - h(Z)$$
$$= h(Y) - \frac{1}{2} \log 2\pi eN.$$  

Since $E[X^2] \leq P$ and $E[Z^2] \leq N$, so $Y = X + Z \rightarrow E(Y^2) \leq P + N$ and

$$h(Y) \leq \frac{1}{2} \log 2\pi e(P + N)$$

where the equality holds iff $Y \sim \mathcal{N}(0, P + N)$ and $X \sim \mathcal{N}(0, P)$.  

Theorem

The capacity of a Gaussian channel with input power constraint $P$ and noise variance $\sigma^2 = N$ is

$$C = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right).$$

This capacity is achieved with $X \sim \mathcal{N}(0, P)$. The proof of this theorem is similar to the proof of the DMC capacity but involves extra steps concerned the power constraint.

Set the SNR $\frac{P}{N} = \gamma$, so it could be written as $C = \frac{1}{2} \log(1 + \gamma)$. In a graph we usually plot the capacity against $\gamma_{dB} = 10 \log_{10} \gamma$.

2 Parallel Gaussian channels

Consider $K$ parallel Gaussian channels with independent noises $Z_i \sim \mathcal{N}(0, N_i)$.

$$Y_k = X_k + Z_k, \quad k = 1 \ldots K$$

The inputs $X_i$ are independent with $Z_j$ and have sum power constraint as

$$\sum_{i=1}^{K} E[X_k^2] = P.$$

Because of independent noises, the transmission rate can be upper-bounded as

$$I(X_1, X_2, \ldots, X_k; Y_1, Y_2, \ldots, Y_k) \leq \sum_{i=1}^{k} I(X_i; Y_i) \leq \frac{1}{2} \sum_{i=1}^{k} \log \left( 1 + \frac{P_i}{N_i} \right).$$

**Question:** How to optimize power allocation?

$$\max \frac{1}{2} \sum_{i=1}^{K} \log \left( 1 + \frac{P_i}{N_i} \right)$$

$$\text{s.t.} \quad \sum P_i = P,$$

$$P_i \geq 0$$

Form the Lagrangian, we have

$$J = \frac{1}{2} \sum \log \left( 1 + \frac{P_i}{N_i} \right) + \lambda \sum P_i$$

$$\frac{\partial J}{\partial P_i} = \frac{1}{2} \frac{1}{P_i + N_i} + \lambda = 0 \quad \rightarrow \quad P_i = -\frac{1}{2\lambda} - N_i.$$  

The optimal power allocation is

$$P_i = (\mu - N_i)^+$$

where $\mu$ is such that $\sum P_i = P$. This is called water-filling.


3 Parallel Gaussian channels with colored noise

In the case of colored noise, \((Z_1, Z_2, \cdots, Z_k) \sim \mathcal{N}(0, K_z)\). (If \(K_z = \text{diag}(N_1, N_1, \ldots, N_k)\) then again we have independent noises as the previous case.)

The optimal input \(X = (X_1, X_2, \cdots, X_k)\) is \(X \sim \mathcal{N}(0, K_x)\). We want to find the optimal \(K_x\), subject to

\[
E(X_1^2) + E(X_2^2) + \cdots + E(X_k^2) \leq P
\]

\[
\Rightarrow \text{tr}(K_x) \leq P.
\]

Since \(Z \sim \mathcal{N}(0, K_z) \rightarrow H(z) = \frac{1}{2} \log (2\pi e |K_z|^n)\). The problem becomes

\[
\max \quad I(X; Y) = \frac{1}{2} \log \left( \frac{\det(K_z + K_x)}{\det(K_z)} \right)
\]

s.t.

\[
\text{tr}(K_x) \leq P,
\]

Let \(K_z = Q\Lambda_z Q^T\) be the eigenvalue decomposition, then \(K_x = Q\Lambda_x Q^T\) is the eigenvalue decomposition of the optimal \(K_x\), where \(\lambda_{x,i} = (\mu - \lambda_{N_i})^+\). In other words, we design the the input signals to be correlated with the same eigenvectors as the noise correlation, and perform water-filling on the eigenvalues of \(K_Z\). This is also the idea in a MIMO channel if both the transmitter and receiver know the channel.

4 Bandlimited Channels

The bandlimited channel is the channel with spectrum \(H(f) = 0\) if \(|f| \geq W\). With sample rate \(2W\), we could reconstruct the signal completely. For each sample, the power constrain becomes \(P_{2W}\).

Assume white noise with Power Spectrum Density(PSD) \(\frac{N_0}{2}\), then the total noise power = \(N_0W\), and noise power per sample is \(\frac{N_0W}{2W} = \frac{N_0}{2}\)

So the capacity per sample is

\[
C_s = \frac{1}{2} \log \left( 1 + \frac{P}{N_0W} \right).
\]

Then the capacity for the bandlimited channel is

\[
C = 2WC_s = W \log \left( 1 + \frac{P}{N_0W} \right).
\]

If the noise is not white but is a stationary Gaussian stochastic process with covariance matrix \(K_Z\), then we perform waterfilling on the eigenvalues of \(K_z\). As the number of samples increases, the density of the eigenvalue of \(K_Z\) tends to the power spectrum of the noise, and waterfilling translates to the spectral domain.