Lecture 3:
Gaussian Channels

1 Scalar Gaussian channel

Consider a discrete-time channel in which the input X_i and output Y_i are samples at time i and are continuous signals. The noise samples Z_i are i.i.d Gaussian, and X_i and Z_i are independent. At each sample, the channel can be expressed as

$$Y_i = X_i + Z_i$$

Without any constraint, the capacity is infinity.

Average power constraint

For each codeword $X^n = (X_1, X_2, \cdots, X_n)$

$$\frac{1}{n}\sum_{i=1}^{n} {X_i}^2 \leq P.$$

We can also write $E[X^2] \leq P$ although this is slightly different as it implies the average power per column of the codebook (at each time sample).

Then using the result from the DMC

$$\begin{aligned} I(X;Y) &= h(Y) - h(Y|X) = h(Y) - h(X+Z|X) \\ &= h(Y) - h(Z|X) = h(Y) - h(Z) \\ &= h(Y) - \frac{1}{2}\log 2\pi e N. \end{aligned}$$

Since $E[X^2] \leq P$ and $E[Z^2] \leq N$, so $Y = X + Z \to E(Y^2) \leq P + N$ and

$$h(Y) \leq \frac{1}{2}\log 2\pi e(P+N)$$

xwhere the equality holds iff $Y \sim \mathcal{N}(0, P+N)$ and $X \sim \mathcal{N}(0, P)$.

Theorem

The capacity of a Gaussian channel with input power constraint P and noise variance $\sigma^2 = N$ is

$$C = \frac{1}{2}\log\left(1+\frac{P}{N}\right).$$

This capacity is achieved with $X \sim \mathcal{N}(0, P)$.

The proof of this theorem is similar to the proof of the DMC capacity but involves extra steps concerned the power constraint.

Set the SNR$= \frac{P}{N} = \gamma$, so it could be written as $C = \frac{1}{2}\log(1+\gamma)$. In a graph we usually plot the capacity against $\gamma_{dB} = 10\log_{10}\gamma$.

2 Parallel Gaussian channels

Consider K parallel Gaussian channels with independent noises $Z_i \sim \mathcal{N}(0, N_i)$.

$$Y_k = X_k + Z_k, \quad k = 1\ldots K$$

The inputs X_i are independent with Z_j and have sum power constraint as

$$\sum_{i=1}^{K} E[X_k^2] = P.$$

Because of indepedent noises, the transmission rate can be upper-bounded as

$$\begin{aligned} I(X_1, X_2, \cdots, X_k; Y_1, Y_2, \cdots, Y_k) &\le \sum_{i=1}^{k} I(X_i; Y_i) \\ &\le \frac{1}{2}\sum_{i=1}^{k}\log\left(1+\frac{P_i}{N_i}\right). \end{aligned}$$

Question: How to optimize power allocation?

$$\begin{aligned} \max \quad & \tfrac{1}{2}\sum \log\left(1+\tfrac{P_i}{N_i}\right) \\ \text{s.t.} \quad & \sum P_i = P, \\ & P_i \ge 0 \end{aligned}$$

Form the Lagrangian, we have

$$\begin{aligned} J &= \frac{1}{2}\sum \log\left(1+\frac{P_i}{N_i}\right) + \lambda \sum P_i \\ \frac{\partial J}{\partial P_i} &= \frac{1}{2}\frac{1}{P_i+N_i} + \lambda = 0 \longrightarrow P_i = -\frac{1}{2\lambda} - N_i. \end{aligned}$$

The optimal power allocation is

$$P_i = (\mu - N_i)^+$$

where μ is such that $\sum P_i = P$. This is called water-filling.

3 Parallel Gaussian channels with colored nosie

In the case of colored noise, $(Z_1, Z_2, \cdots, Z_k) \sim \mathcal{N}(0, K_z)$. (If $K_z = \text{diag}(N_1, N_1, \ldots, N_k)$ then again we have independent noises as the previouse case.)

The optimal input $X = (X_1, X_2, \cdots, X_k)$ is $X \sim \mathcal{N}(0, K_x)$. We want to find the optimal K_x, subject to

$$\begin{aligned} E(X_1^2) + E(X_2^2) + \cdots + E(X_k^2) &\leq P \\ \Rightarrow \text{tr}(K_x) &\leq P. \end{aligned}$$

Since $Z \sim \mathcal{N}(0, K_z) \rightarrow H(z) = \frac{1}{2}\log\left(2\pi e |K_z|^n\right)$. The problem becomes

$$\begin{aligned} \max \quad & I(X;Y) = \frac{1}{2}\log\left(\frac{\det(K_z + K_x)}{\det(K_z)}\right) \\ \text{s.t.} \quad & \text{tr}(K_x) \leq P, \end{aligned}$$

Let $K_z = Q\Lambda_z Q^T$ be the eigenvalue decomposition, then $K_x = Q\Lambda_z Q^T$ is the eigenvalue decomposition of the optimal K_x, where $\lambda_{x_i} = (\mu - \lambda_{N_i})^+$. In other words, we design the the input signals to be correlated with the same eigenvectors as the noise correlation, and perform water-filling on the eigenvalues of K_Z. This is also the idea in a MIMO channel if both the transmitter and receiver know the channel.

4 Bandlimited Channels

The bandlimited channel is the channel with spectrum $H(f) = 0$ if $|f| \geq W$. With sample rate 2W, we could reconstruct the signal completely. For each sample, the power constrain becomes $\frac{P}{2W}$. Assume white noise with Power Spectrum Density(PSD) $\frac{N_0}{2}$, then the total noise power $= N_0 W$, and noise power per sample is $\frac{N_0 W}{2W} = \frac{N_0}{2}$

So the capacity per sample is

$$C_s = \frac{1}{2}\log\left(1 + \frac{P}{N_0 W}\right).$$

Then the capacity for the bandlimited channel is

$$C = 2WC_s = W\log\left(1 + \frac{P}{N_0 W}\right).$$

If the noise is not white but is a stationary Gaussian stochastic process with covariance matrix K_Z, then we perform waterfilling on the eigenvalues of K_z. As the number of samples increases, the density of the eigenvalue of K_Z tends to the power spectrum of the noise, and waterfilling translates to the spectral domain.