Statistically-based Sequential Detection of Buried Mines from Array Ground Penetrating Radar Data

Xiaoyin Xu, Eric L. Miller, and Carey M. Rappaport
Center for Electromagnetics Research
Dept. of Electrical and Computer Engineering
Northeastern University, Boston, MA 02115
Telephone: (617) 373-8386
Telefax: (617) 373-8627

ABSTRACT
We consider the problem of detecting and localizing buried landmines from a ground penetrating radar (GPR) array. A simplified, ray-optics-based physical model for time domain GPR returns is presented. Under this model in the absence of an object from the field of view of the array, there exist well defined symmetries in the structure of the radar returns. In particular, for a bistatic system composed of one length $M$ transmit array and a second length $M$ array of receivers, we identify $M$ subsets of signals from the $M^2$ total transmitter/receiver pairs such that the mean value of the signals within each subset should be the same when no object is present. This relationship then forms the basis for a modified Hotelling’s $T^2$-test to detect the presence of objects when there is noise in the signal. Simulation results demonstrate the validity of these methods.

1. INTRODUCTION
Mines kill or maim hundreds of people every week, mostly innocent and defenseless civilians. Among the various demining methods developed in recent years, ground penetrating radar (GPR) holds substantial promise because of its sensitivity to non-metallic, plastic objects which traditional metal detectors are largely incapable of finding. Nevertheless, using GPR to detect and localize mines is both difficult and complex. One key problem is the rejection of interference caused by the signal arising from scatter off the air-earth interface. Both the magnitude of the ground bounce and its timing are such that they can easily swamp the relatively small signal arising from the interaction of the transmitted GPR waveform with the buried mine. To help overcome this difficulty, we consider the use of a GPR array to provide a richer and more diverse set of data thereby making accurate detection and localization possible in circumstances where a single GPR is unable to perform well.

In this paper we discuss a method to detect mines which exploits both the physics of the problem as well as the geometry of the array system. We assume that the GPR array is deployed as shown in Fig. 3 with one linear array of transmitters and a second array of receivers both traveling down a track. As described in Section 3, the geometric symmetry inherent in this configuration introduces a “statistical symmetry” in the received signals. In particular, this symmetry is preserved precisely when there is no mine. On the other hand, the presence of a mine will break this symmetry and therefore provide information as to the existence of an object. Based on this relationship, we develop a statistical test of homogeneity to ascertain the presence of an object in the field of view of the array.

The organization of this paper is as follows. Section 2 introduces the physical model of GPR signals and the configuration of the GPR array. In Section 3 we present the detection algorithm, using a modified

Other author information: XX: xxu@cdsp.neu.edu, ELM: elmiller@ece.neu.edu, CR: rappaport@neu.edu. This work was supported by the Army Research Office Demining MURI under Grant DAAG55-97-1-0013
form of Hotelling’s $T^2$-test. Examples are given in Section 4 to demonstrate validity of the algorithm. Section 5 summarizes present work and points out future research direction.

2. PHYSICAL MODEL AND PROBLEM FORMULATION

2.1. A Single GPR System

To detect and localize mines, a ground penetrating radar array is implemented. Fig. 1 shows a typical single GPR system with the signals it generates. In this paper we assume a simplified model where the signal seen by the receiver is composed of at most two components. The first signal is the reflected signal from the ground and is always present in the data. The second component (if it exists) is the reflected contribution from an object in the field of view of the array.

![Figure 1. Schematic drawing of a single GPR, transmitter and receiver.](image)

The received signal, $\phi(t)$, is taken to be the sum of delayed and attenuated versions of two “template” signals indicating the nominal behavior of the ground bounce signal and the nominal behavior of a signal arising from scattering from a mine. Mathematically we have

$$
\phi(t) = a \psi_g(t - \tau_g) + b \psi_m(t - \tau_m)
$$

where $\psi_g$ and $\psi_m$ are the nominal ground bounce and mine reflected signal, $a$ and $b$ are attenuation factors, $\tau_g$ is the delay of the ground reflection, and $\tau_m$ is the delay of the mine signal. Note that if no mine is present, $\phi$ is just equal to the first term of (1).

To find the delays and the attenuation factors we assume that the propagation of the signal from the transmitter to the receiver can be described using a ray-optics-type model shown in Fig. 1. That is, the ground bounce is composed of signal reflected from the interface at the specular point midway between the transmitter and receiver while the four-part path of the mine component of the signal can be determined via the judicious use of Snell’s law.

To begin, the $\tau_g$ and $\tau_m$ are determined by the travel time of two-way paths and can be calculated as

$$
\text{Delay} = \frac{\text{2-way path length}}{\text{velocity of the wave}}.
$$

To find $\tau_g$ and $\tau_m$, we need to locate reflecting point and refracting point shown in Fig. 2. Let media 1 be air and media 2 be soil, with electric permittivity $\epsilon_0$ and $\epsilon_1$, respectively, Fig. 2(a), the reflecting point on the boundary between two points $(x_1, y_1)$ and $(x_2, y_1)$ in media 1 is simply the mid-point $(x_4, 0)$, where $x_4 = \frac{x_1 + x_2}{2}$. 

Figure 2. Geometries for determining (a) the reflecting point and (b) the refracting point, $\epsilon_1 > \epsilon_0$.

For the refracting point, according to Snell’s law, for a source located at $(x_1, y_1)$ in media 1 and target at $(x_3, y_3)$ in media 2, the refracted ray from source to target must intersect the boundary at a point $(x_5, 0)$, Fig. 2(b), such that

$$\frac{Re\{\epsilon_1\}}{\epsilon_0} = \frac{(x_1 - x_5)^2}{(x_1 - x_3)^2 + y_1^2} \times \frac{(x_3 - x_5)^2 + y_3^2}{(x_3 - x_3)^2 + y_3^2},$$

(3)

Solution of this quartic equation has four roots. By Fermat’s principle, which states that of all possible paths joining two given points on a wave path, the wave path has actual least travel time, we can discard three physically impossible roots and retain the true refracting point. Once the reflecting point and the refracting point are established, the delay $\tau_g$ and $\tau_m$ can be found as,

$$\tau_g = \frac{2\sqrt{(x_1 - x_4)^2 + y_1^2}}{c}$$

$$\tau_m = \frac{2\sqrt{(x_1 - x_5)^2 + y_1^2} + 2\sqrt{(x_3 - x_5)^2 + y_3^2}}{c/Re\sqrt{\epsilon_1}}$$

(4)

(5)

where $c$ is the speed of light in air and $Re\sqrt{\epsilon_1}$ is the speed of the wave in soil.

In addition to the time delays, the received signal $\phi(t)$ has an amplitude reduction caused by propagation through the soil as well as geometric spreading as it traverses both the air and the earth. In soil, the wave attenuates exponentially with the distance it travels, $e^{-\alpha_s d}$. The quantity $\alpha_s$ is the attenuation constant of the soil which is related to the conductivity and permittivity of the medium while $d$ is the distance the wave travels in the earth. We assume geometric spreading results in an inverse path length amplitude reduction. Referring to the setup of Fig. 2, then we have the overall amplitude reduction factors given by

$$a = \frac{1}{2\sqrt{(x_1 - x_4)^2 + y_1^2}}$$

and

$$b = \left(\frac{e^{-\alpha_s \sqrt{(x_3 - x_5)^2 + y_3^2}}}{\sqrt{(x_1 - x_4)^2 + y_1^2} + \sqrt{(x_3 - x_5)^2 + y_3^2}}\right)^2.$$

2.2. GPR Array

In this work, the GPR array is assumed to consist of $M$ pairs of transmitters and receivers. Data are collected by the GPR array as it travels step by step down track, Fig. 3. At each stop of the array, $M^2$ signals (time-traces) are collected; one for each transmitter/receiver pair.
Using the model developed in the previous section and assuming that the GPR array is at its $k$-th stop, the signal seen at receiver $j$ due to input from transmitter $i$ is written as

$$\phi_{ij}^k(t) = a_{ij} \psi_g(t - \tau_{g,ij}) + b_{ij} \psi_m(t - \tau_{m,ij}), \quad i, j = 1, \ldots, M, \ k = 1, \ldots, K.$$  \hspace{1cm} (6)

Note that $a$ and $\tau_g$ do not depend on $k$, the down-track GPR array position. This is easy to understand because the ground-reflected signal only depends on the relative position of transmitter and receiver. To simplify matters, in the future we use the following shorthand

$$g_{ij}^k = a_{ij} \psi_g(t - \tau_{g,ij}) \quad s_{ij}^k = b_{ij} \psi_m(t - \tau_{m,ij}).$$

Each $\phi_{ij}^k(t)$ is densely sampled $P$ times over a time interval. The interval is chosen to be long enough to embrace both ground bounce and mine signal. When no ambiguity will arise, we refer to the vector of samples, $\omega_{ij}^k$, rather than the temporal signal, $\phi_{ij}^k(t)$ with a similar interpretation holding for $g_{ij}$ and $s_{ij}$. Note $\omega_{ij}^k$, $g_{ij}$, and $s_{ij}^k$ are column vectors of size $P$.

For a given location of the GPR array, to detect mines, we carry out a binary hypothesis test. Under the null hypothesis, $H_0$, the received signal $\omega_{ij}^k$ is comprised of ground bounce $g_{ij}$ plus measurement noise, which is assumed to be a white Gaussian vector, $w \sim N(0, \sigma^2 I)$. Under the alternate hypothesis, $H_1$, $\omega_{ij}^k$ consists of ground bounce, noise, and mine signal, $s_{ij}^k$. Mathematically we have

$$H_0^k : \quad \omega_{ij}^k = g_{ij} + w$$

$$H_1^k : \quad \omega_{ij}^k = g_{ij} + s_{ij}^k + w.$$  \hspace{1cm} (7)

Our processing method is based on the observation that under $H_0$, the $M^2$ received signal should display certain symmetries, as illustrated by Fig. 4. For example if no mine is present then $\omega_{ij}$ should be “statistically equal” to $\omega_{i+1,j}$, $\omega_{i+2,j}$, $\omega_{i+3,j}$, and $\omega_{i+4,j}$ because the ground bounce in each case depends only on the relative spacing of the sensors which is identical for these six pairs. Similarly, $\omega_{j+1}$ should be statistically equal to $\omega_{j+1,i}$, and so on. By “statistically equal” we mean that any variations in these signal are caused by random sensor noise. In other words, signals from these sets will, on average, possess the same means with some variability (variance) caused by the noise. Thus statistical tests designed to determine homogeneity of a population (i.e. equality of mean vectors) can be used to test whether an object is present (lack of homogeneity) or absent (all the data vectors are about the same). Finally, for the $M^2$ received signals it is not hard to show that there are only $M$ sets of statistically different signals because of this symmetry. Note the $M$ sets of signals are not of equal size, some sets consist of more signals than the others.

*Here we are assuming that the ground is locally flat over the extent of the sensing system. Extension of the results in this work to smoothly changing ground is an area of current work.
From the above discussion, we know that if a mine is present, the symmetry of received signals will be disrupted. Therefore, to detect mines, we can look for asymmetry of received signals. Basically, our approach is to sequentially detect any changes in the means of the received signals. For the purpose of illustration, we use $M = 4$ pairs of transmitter and receiver. At each stop $k$ of the GPR array, we then have 4 sets of statistically different signals, $S_1 = \{\phi_1^k, \phi_2^k, \phi_3^k, \phi_4^k\}$, $S_2 = \{\phi_1^k, \phi_2^k, \phi_3^k, \phi_4^k, \phi_5^k, \phi_6^k\}$, $S_3 = \{\phi_3^k, \phi_3^k, \phi_4^k, \phi_4^k\}$, and $S_4 = \{\phi_4^k, \phi_4^k\}$.

### 3. INFORMATION EXTRACTION ALGORITHM

As the problem currently stands, we do not assume that the ground bounce signals are known. In fact, a simple transformation of the data allows us to perform the test without ever having to know the ground bounces or estimate them. To see this, we begin by forming the collection of all pairwise differences of the signals within the set under consideration. For $S_1$ we get the set $D_1$ defined as

$$D_1 = \{\phi_1 - \phi_2, \phi_1 - \phi_3, \phi_1 - \phi_4, \phi_2 - \phi_3, \phi_2 - \phi_4, \phi_3 - \phi_4\} \equiv \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}.$$

Testing $H_0$ using the original set of $\phi_i$ signals then is equivalent to testing the following hypothesis using the $\delta_i$ vectors:

$$H_0:\ \delta_i \text{ in the set are from same multivariate Gaussian distribution, with equal and known covariance matrices and means equal to zero}$$

### 3.1. Homogeneity Test of Single Set

Here we discuss the homogeneity test of a single set, say $S_1 = \{\phi_1^k, \phi_2^k, \phi_3^k, \phi_4^k\}$. We consider the acceptance test of the hypothesis

$$H_0:\ \phi_i^k \text{ in the set are from same multivariate Gaussian distribution, with equal and known covariance matrices, and identical, unknown means, } i = 1, \ldots, 4.$$
To test this equality of means, we carry out a Hotelling’s $T^2$-test.\footnote{As described more fully in [5], this test amounts to the following comparison}

$$
\xi_1 = \sum_{n=1}^{6} \frac{1}{2} \frac{1}{\sigma^2} \mathbf{R}^{-1} \mathbf{\hat{z}}_n \leq \text{threshold}
$$

with $\mathbf{R} (= \sigma^2 \mathbf{I})$ is the covariance matrix of the measurement noise and threshold is chosen to ensure an \textit{a priori} specified probability of false acceptance. The hypothesis is accepted if the left hand side is less than or equal to threshold and rejected otherwise. Noting that $\mathbf{R}$ is diagonal, we write $\xi_1$ as

$$
\xi_1 = \frac{1}{2\sigma^2} \mathbf{1}_4^T \mathbf{A}_1 \mathbf{1}_4 = \frac{1}{2\sigma^2} \left[ \begin{array}{ccc}
\phi_1^T \\
\phi_2^T \\
\phi_3^T \\
\phi_4^T \\
\end{array} \right] \left[ \begin{array}{cccc}
3 \mathbf{I} & -\mathbf{I} & -\mathbf{I} & -\mathbf{I} \\
-\mathbf{I} & 3 \mathbf{I} & -\mathbf{I} & -\mathbf{I} \\
-\mathbf{I} & -\mathbf{I} & 3 \mathbf{I} & -\mathbf{I} \\
-\mathbf{I} & -\mathbf{I} & -\mathbf{I} & 3 \mathbf{I} \\
\end{array} \right] \left[ \begin{array}{c}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4 \\
\end{array} \right] .
$$

For a set $S_i$ of size $N_i$, it is easy to show that $\mathbf{A}_i$ is an $N_i \times N_i$ block matrix. Its block diagonal elements are $(N_i - 1) \mathbf{I}_{P \times P}$ and the off diagonal elements are $- \mathbf{I}_{P \times P}$.

### 3.2. Homogeneity Test of Multiple Sets

For the application of interest in this paper, we have multiple sets (four for our sample system) for which we wish to test homogeneity. Here we adopt a simple, generalized form of the $T^2$ test in which the four $\xi_i$’s are weighted and added to yield one final test statistic,

$$
\xi = \xi_1 \eta_1 + \xi_2 \eta_2 + \xi_3 \eta_3 + \xi_4 \eta_4 .
$$

The $\eta_i$ are weight factors, defined by

$$
\eta_i = \frac{1}{l_i} \left( \sum_{j=1}^{4} l_j^{-1} \right)^{-1}
$$

where $l_j$ are the path lengths of ground bounces for set $S_j$. More will be said about this weight factor in Section 3.3. Stacking all the 16 signals $\phi_{ij}$ to form a long column vector, we obtain a new vector

$$
\mathbf{\Phi}^T = \left[ \mathbf{\Phi}_1^T \mathbf{\Phi}_2^T \mathbf{\Phi}_3^T \mathbf{\Phi}_4^T \right]
$$

and

$$
\xi = \frac{1}{2\sigma^2} \mathbf{\Phi}^T \mathbf{A} \mathbf{\Phi} = \frac{1}{2\sigma^2} \left[ \begin{array}{c}
\mathbf{A}_1 \eta_1 \\
0 \\
0 \\
0 \\
\end{array} \right] \left[ \begin{array}{c}
\mathbf{A}_2 \eta_2 \\
0 \\
0 \\
0 \\
\end{array} \right] \left[ \begin{array}{c}
\mathbf{A}_3 \eta_3 \\
0 \\
0 \\
0 \\
\end{array} \right] \left[ \begin{array}{c}
\mathbf{A}_4 \eta_4 \\
0 \\
0 \\
0 \\
\end{array} \right] .
$$

Matrix $\mathbf{A}$ is a block diagonal matrix. Obviously, $\mathbf{A}$ is symmetric. Note $\mathbf{A}$ is positive semidefinite by its buildup. According to Mathai,\footnote{Most two moments of $\xi$ are given by,}

$$
\mu_\xi = (6 \eta_1 + 15 \eta_2 + 6 \eta_3 + \eta_4) P
$$

$$
\sigma_\xi^2 = (24 \eta_1^2 + 90 \eta_2^2 + 24 \eta_3^2 + 2 \eta_4^2) P ,
$$

\(\text{ denotes the transpose.}\)
It has been shown that $\xi$ asymptotically has a Gaussian distribution\(^6\) for large $NP$, i.e., $\xi \sim N(\mu_\xi, \sigma_\xi^2)$. The generalized $T^2$-test amounts to $\xi \leq \alpha$ where $\alpha$ is the threshold chosen to ensure an $a \ priori$ specified probability $P_{fa}$ of false acceptance. The $\alpha$ is determined by

$$P_{fa} = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi\sigma_\xi^2}} e^{-\frac{(\xi-\mu_\xi)^2}{2\sigma_\xi^2}} d\xi, \quad (14)$$

### 3.3. Sequential Detection

Now we consider the actual GPR array operation as the array moves down-track. At each stop $k$, we calculate a corresponding $\xi^k$. When there is no mine in the field of view of the array, we record $\xi^k$ only. That is, $\nu^k = \xi^k$ under hypothesis $H_0$. When there is a mine, the mine signal $s_j^k$ adds to the ground bounce. Passing $s_j^k$ to the generalized Hotelling’s $T^2$-test produces an output signal, denoted by $\zeta^k$. So the problem is to detect signal $\zeta^k$, given the observed signal sequence $\nu^k$ and known “noise” $\xi^k$, using the additive noise model $\nu^k = \xi^k + \zeta^k$ under the alternate hypothesis $H_1$. Though $\zeta^k$ is unknown and changes with $k$, it is always positive because of the very nature of quadratic form of the generalized $T^2$-test. Fig. 5 shows separate $\xi^k$ and $\xi^k + \zeta^k$. Signals on the left side of Fig. 5 are typical “noise” sequences, taken from four different sets of transmitter-receiver combinations. Signals on the right side are “noise” and mine signals, corresponding to the respective sets of transmitter-receiver combinations. It is seen that $\zeta^k$ is much weaker in set $S_3$ and $S_4$. This observation can be explained by the increased attenuation associated with the longer distances the mine signals in sets $S_3$ and $S_4$ travel. For this reason, we introduced the weight factors in Eq. 11.

When the mine is buried deep, the mine signal attenuates exponentially as explained in Section 2.1. Detecting this unknown low power signal can be aided quite a bit using sequential detection methods which retain information from previous scans to improve the SNR. We therefore employ a sequential detector as the GPR array moves down track. At each stop $k$, the detector makes one of two decisions\(^7\): (1) Hypothesis $H_0$ is true, no mine signal is present, (2) Reject $H_0$. Because $\zeta^k$ causes a positive displacement of the mean of $\nu^k$, we choose a running average of $\nu^k$ as a statistical test\(^8\)

$$g^k = \frac{1}{N} \sum_{j=k-N+1}^{k} \nu^j, \quad k = N, N + 1, \cdots, K \quad (15)$$

and make a decision by checking $g^k \leq \beta$ where $\beta$ is a threshold. At each stop $k$, $g^k$ is compared to the threshold to make a declaration of mine presence. Fig. 6 illustrates relationship between threshold settings and declaration of mine presence as the GPR arrays move down-track. Each filled dot indicates a location of the array where we say that a mine is present. Such a declaration is made when $g^k$ is above the threshold for consecutive 5 stops of $k$. A lower threshold allows us to make early declaration of mine. The disadvantage is a high false acceptance rate. The false acceptance probability $P_{fa}$ equals the probability under hypothesis $H_0$ that the $g^k$ crosses the threshold. The detection probability $P_d$ equals to $1 - P_{fa}$. By virtue of the generalized $T^2$-test, $\nu^k$ are statistically independent. Hence, $g^k$ has the Gaussian distribution, $N(\mu_\xi, \frac{1}{N}\sigma_\xi^2)$. Thus, in principle, for a given $P_{fa}$, $\beta$ can be determined similarly as in Eq. 14.

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1. See appendix.
Figure 5. First row of figures corresponds to the $T^2$ statistic for set $S_1$, the left figure is “noise” only, the right figure has both “noise” and mine reflected signal, second row of figures corresponds to set $S_2$, third row of figures corresponds to set $S_3$, and, fourth row of figures corresponds to set $S_4$.

Figure 6. Detection results versus prescribed false alarm probability, $P_{fa}$. Note that lower $P_{fa}$ results in slower detection while a higher allowable false alarm rate allows for detection prior to the array passing over the mine.
In this section we consider some examples of the above procedure. Monte-Carlo runs were used to determine the probability of detection and probability of false-alarm for various noise levels and depth of buried mines.

In this work, the signal to interference plus noise ratio is defined as

\[
\text{SINR} = 10 \log_{10} \frac{\mathbf{s}^T \mathbf{s}}{q^T q + P \sigma^2}.
\]

In all cases, we generate synthetic data with an object located around the 50th stops of the GPR array and buried 10 cm underground. The GPR array is composed of four pairs of transmitters and receivers evenly spaced along a baseline width of 80 cm. A transmitter and its corresponding receiver (e.g. transmitter 1 and receiver 1) are 20 cm apart. The GPR array is 40 cm above the ground.

For simplicity we assume that the nominal ground bounce and mine-bounce signal take the form of a second derivative of a Gaussian shown in Fig. 7(a). Current work in our group is aimed at developing more sophisticated models for these signals. Fig. 7(b) and (c) show the received signals of two pairs of T/R combinations. Because of the domination of the ground bounce, it is difficult to see any mine signature. Even after pairwise subtraction, the mine signal can hardly be observed in Fig. 7(d). But the generalized \(T^2\)-test can pick up this difference and declare a mine.

Fig. 8(a) shows the receiver operating characteristics of detecting a mine buried 10 cm underground. Fig. 8(b) shows two ROC’s for mines buried at different depth under the same noise power. Because of the fast attenuation in soil, the deeper buried mine has a significantly smaller SINR thereby leading to the degradation in performance.

5. CONCLUSIONS AND FUTURE WORK

In this paper, we discuss a statistical approach to detect mines using a GPR array. Exploiting the geometric symmetry of GPR array setting, we have looked into tests for statistical homogeneity of GPR returns as a tool for performing detection. Basically, by treating the received signal as a multivariate Gaussian distribution we test its statistical homogeneity using a generalized, sequential Hotelling’s \(T^2\)-test.

Future work will take into consideration of rough ground-air interface and time-delay estimation to actually localize detected mines. An assumption of this work is that the soil conditions are known. Sensitivity analysis involving mismatch in soil parameters will be valuable to apply this method to realistic scenario. Moreover, the additive white Gaussian noise mode will be lifted as we explore issues associated with the modeling of clutter, the incorporation of these models in to our processing, and the development of test which are robust (or invariant) to uncertainty in these models.

6. APPENDIX

From Mathai’s work, for \(X \sim N(\mu, \Sigma)\), the first two moments of quadratic \(X^T A X\) can be found by

\[
E[\xi] = tr(A \Sigma) + \mu^T A \mu \\
\text{Var}[\xi] = 2 tr(A \Sigma)^2 + 4 \mu^T A \Sigma A \mu.
\]

For the generalized \(T^2\)-test, we have

\[
\mu^T = \left[\mu_{1}^T, \cdots, \mu_{14}^T, \mu_{2}^T, \cdots, \mu_{34}^T, \mu_{3}^T, \cdots, \mu_{24}^T, \mu_{4}^T, \mu_{11}^T\right]
\]
Figure 7. Received signal and time traces.
Figure 8. Receiver operating characteristics.

where

\[
\mu_1 = \cdots = \mu_{44} \equiv \mu^{(1)}_1 \quad \mu_{12} = \cdots = \mu_{24} \equiv \mu^{(2)}_1 \quad \mu_{13} = \cdots = \mu_{23} \equiv \mu^{(3)}_1 \quad \mu_{14} = \mu_{41} \equiv \mu^{(4)}_1.
\]  

(19)

Next, we note that \(4\mu^T A \Sigma A \mu = 0\) under the condition of Eq. 19,

\[
4\mu^T A \Sigma A \mu = 4 \sum_{i=1}^{4} \left[ \mu^{(i)}_1 \cdots \mu^{(i)}_N \right] A_i \Sigma A_i \left[ \mu^{(i)}_1 \cdots \mu^{(i)}_N \right] \]

(20)

where

\[
\begin{bmatrix} \mu^{(i)}_1 \cdots \mu^{(i)}_N \end{bmatrix} A_i = 0
\]  

(21)

because of the semidefiniteness of \(A_i\). Similarly, \(\mu^T A \mu = 0\). The mean of \(\xi\) is then given by \(\frac{1}{2\sigma^2} [tr(A \Sigma)]\). For \(A = \text{block diag} \{A_1 \eta_1, A_2 \eta_2, A_3 \eta_3, A_4 \eta_4\}\) we have

\[
E[\xi] = \frac{1}{2\sigma^2} tr(A \Sigma) = \frac{1}{2} \sum_{i=1}^{4} N_i (N_i - 1) P \eta_i.
\]  

(22)

Additionally \(A A = \text{block diag} \{A_1 A_1 \eta_1^2, A_2 A_2 \eta_2^2, A_3 A_3 \eta_3^2, A_4 A_4 \eta_4^2\}\) where

\[
A_i A_i = \begin{bmatrix}
(N_i^2 - N_i) I & -N_i I & \cdots & -N_i I \\
-N_i I & (N_i^2 - N_i) I & \cdots & -N_i I \\
\vdots & \vdots & \ddots & \vdots \\
-N_i I & -N_i I & \cdots & (N_i^2 - N_i) I
\end{bmatrix}
\]  

(23)
Here $I$ is of size $P \times P$ and $A_i$ is of size $N_i P \times N_i P$. We then have

$$tr(A_i)^2 = (N_i^2 - N_i)N_i P \quad tr(A)^2 = \sum_{i=1}^{4} tr(A_i)^2 = \sum_{i=1}^{4} (N_i^2 - N_i)N_i P \eta_i^2,$$

so that $Var[\xi] = \frac{1}{4\pi} 2 tr(A \Sigma)^2 = \frac{1}{2} \sum_{i=1}^{4} (N_i^2 - N_i)N_i P \eta_i^2$.

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