Topic 3: Operations on a random variable

- Function of a random variable
- Transform methods
- Generation of random variables

Function of a random variable

Let $g(x)$ be a real-value function of the real line, $g : \mathbb{R} \to \mathbb{R}$. Let X be a random variable and let

$$Y = g(X)$$

then Y is also a random variable.

- The distribution of Y can be derived from the distribution of X.
- Derived cdf:
$$F_Y(y) = P[Y \leq y] = P[x \mid g(x) \leq y]$$
- If X is a discrete r.v., then Y is also discrete with pmf
$$p_Y(y_k) = \sum_{x_j : g(x_j) = y_k} p_X(x_j)$$

Derived density – Specific functions

Let us consider two specific functions $g(x)$ first, then study the general principle.

- Linear function:

$$Y = aX + b \quad \Rightarrow \quad f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

  - Example: A linear function of a Gaussian r.v. is again a Gaussian random variable.

$$X \sim \mathcal{N}(\mu, \sigma^2) \quad \Rightarrow \quad Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

- Quadratic function:

$$Y = X^2 \quad \Rightarrow \quad f_Y(y) = \frac{f_X\left(\sqrt{y}\right)}{2\sqrt{y}} + \frac{f_X\left(-\sqrt{y}\right)}{2\sqrt{y}} \ , \quad y \geq 0$$

  - Example: Square of a Gaussian r.v. is a Chi-square r.v.

$$X \sim \mathcal{N}(0,1) \quad \Rightarrow \quad Y \sim \mathcal{X}_2^2 \ , \ f_Y(y) = \frac{e^{-y/2}}{\sqrt{2\pi y}} \ , \quad y \geq 0$$

Derived density – The general case

If the equation $g(x) = y$ has n solutions $\{x_1, \ldots, x_n\}$, then

$$f_Y(y) = \sum_{k=1}^{n} f_X(x) \left|\frac{dx}{dy}\right|\Bigg|_{x=x_k} = \sum_{k=1}^{n} \frac{f_X(x_k)}{|g'(x_k)|}$$

where $g'(x_k)$ is the derivative of $g(x)$ evaluated at x_k. Note that each x_k is a function of y.

- Example: $Y = \cos(X)$, where $X \sim \mathcal{U}[-\pi, \pi]$.
  For $-1 \leq y \leq 1$, the equation $y = \cos(x)$ has two solutions in $[-\pi, \pi]$

$$x_1 = \cos^{-1}(y) \quad \text{and} \quad x_2 = 2\pi - x_1.$$

  Calculate the derivatives of y at these points as

$$\begin{aligned} \frac{dy}{dx}|_{x=x_1} &= -\sin(x_1) = -\sin\left(\cos^{-1}(y)\right) = -\sqrt{1-y^2} \\ \frac{dy}{dx}|_{x=x_2} &= -\sin(2\pi - x_1) = \sin(x_1) = \sqrt{1-y^2} \end{aligned}$$

Since $f_X(x) = \frac{1}{2\pi}$, we have

$$f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}} \quad \text{for } -1 \le y \le 1$$

Y is said to have *arcsine distribution.*

Transform methods

- The *characteristic function* is defined as

$$\Phi_X(\omega) = E\left[e^{j\omega X}\right] = \int_{-\infty}^{+\infty} f_X(x)e^{j\omega x}dx \quad \text{for real } \omega, \;\; -\infty < \omega < \infty$$

It is the *Fourier transform* of $f_X(x)$ (with the sign of ω reversed).
Similarly for discrete random variables

$$\Phi_X(\omega) = E\left[e^{j\omega X}\right] = \sum_{k=-\infty}^{\infty} p_X(x_k)e^{j\omega x_k}$$

– Properties
  o The characteristic function always exists.
  o Its maximum magnitude is 1 at $\omega = 0$

$$|\Phi_X(\omega)| \le \Phi_X(0) = 1$$

– Inverse transform

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_X(\omega)e^{-j\omega x}d\omega$$

– Examples: Find $\Phi_X(\omega)$ for exponential, Gaussian RVs.

- The *moment theorem*: All the moments of X, if exist, can be calculated from $\Phi_X(\omega)$ as

$$E[X^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_X(\omega)\bigg|_{\omega=0}$$

  - To show this, expand $e^{j\omega x}$ in a power series and write

$$\Phi_X(\omega) = 1 + j\omega E[X] + \frac{(j\omega)^2 E[X^2]}{2!} + \cdots \frac{(j\omega)^n E[X^n]}{n!} + \cdots$$

    Then differentiate this expression wrt ω and evaluate at $\omega = 0$.

- The *moment generating* function is defined as

$$M_X(s) = E\left[e^{sX}\right] = \int_{-\infty}^{\infty} f_X(x) e^{sx}\, dx \quad \text{for real } s, \quad -\infty < s < \infty$$

  This is the *Laplace transform* of $f_X(x)$ with the sign of s reversed.

  - The moments can be obtained from $M_X(x)$ as

$$E[X^n] = \frac{d^n}{ds^n} M_X(s)\bigg|_{s=0}$$

  - A drawback is that $M_X(s)$ may not exist for all distributions and all values of s (but it needs to exist only around $s = 0$).

- The *probability generating* function for a discrete r.v. N with integer values is defined as

$$G_N(z) = E\left[z^N\right] = \sum_{k=0}^{+\infty} p_N(k) z^k$$

  This is the z *transform* of $p_N(k)$ (again with the sign of the exponent reversed).

  - The pmf of N can be calculated as

$$p_N(k) = \frac{1}{k!} \frac{d^k}{dz^k} G_N(z)\bigg|_{z=0}$$

  - The mean and variance of N are

$$\begin{aligned} E[N] &= G'_N(1) \\ \text{var}(N) &= G''_N(1) + G'_N(1) - \left(G'_N(1)\right)^2 \end{aligned}$$

  - Examples: Find $G_N(z)$ for Bernoulli, binomial, and Poisson RVs.

- Why so many transforms?

Generation of random variables

- We can generate a $U[0,1]$ r.v. from any continuous r.v.
- Vice-versa, we can generate any distribution from a $U[0,1]$ r.v.
- Generating the uniform $U[0,1]$ r.v. from a distribution $F_X(x)$
  - Given X with distribution $F_X(x)$, we want to find a function $g(.)$ such that $U = g(X)$ is uniform in $[0,1]$.
  - It turns out that $g(X)$ is given precisely by $F(X)$

$$U = F(X) \;\text{ has }\; F_U(u) = u \;\text{ for }\; 0 \le u \le 1$$

  - Assuming $F_X(x)$ has an inverse F^{-1}, then

$$F_U(u) = P[U \le u] = P[F(X) \le u] = P[X \le F^{-1}(u)] = F(F^{-1}(u)) = u$$

    Note: $F_X(x)$ does not need to be invertible for the result to apply.
- Generating a random variable Y with cdf $F_Y(y)$ from a $U[0,1]$ r.v. U.
  - Let $Y = F^{-1}(U)$, then

$$F_Y(y) = P[Y \le y] = P[F^{-1}(U) \le y] = P[U \le F(y)] = F(y)$$

  - Examples: Generate the Gaussian and exponential RVs from $U[0,1]$.

Random number generators in computers

How to generate a $U[0,1]$ r.v. in the computer?

- There are uncountably infinite number of points in $[0,1]$, but computers only have *finite precision.*
  $\Rightarrow$ Need to generate equiprobable numbers from a finite set $\{0, 1, \ldots, M-1\}$.
- A naive method: Perform random experiments, such as flipping a coin $\log_2(M)$ times or drawing a ball from those numbered 1 to M. This method requires a large storage space as the sequence grows.
- The *power residue method*:

$$Z_k = \alpha Z_{k-1} \mod M$$

  where M is a large prime number (or an integer power of prime), and α is an integer carefully chosen between 1 and M.
  - The sequence generated is called *pseudo-random* since it is periodic with maximum period M. Hence we want a large M.
  - The starting point Z_0 of a sequence is called the *seed.*

  Example: $\alpha = 7^5$ and $M = 2^{31} - 1$.

Computer generation of random variables

Suppose we want to generate a r.v. X with cdf $F_X(x)$ (or pdf $f_X(x)$).

- The transformation method:
  - Generate U uniform in $[0,1]$.
  - Set $X = F_X^{-1}(U)$.
- The rejection method (general): Generating r.v. $X \sim f_X(x)$
  1. Generate X_1 with an easy pdf $f_W(x)$. Define

     $$B(x) = K f_W(x) \geq f_X(x)$$

     for some constant $K > 1$.
  2. Generate Y uniform in $[0, B(X_1)]$.
  3. If $Y \leq f_X(X_1)$, then output $X = X_1$;
     else reject X_1 and return to step 1.