Gaussian Channel *

1 Definitions

Definition (Gaussian channel) Discrete-time channel with input $X_i$, noise $Z_i$, and output $Y_i$ at time $i$. This is

\[ Y_i = X_i + Z_i, \]

where the noise $Z_i$ is drawn i.i.d. from $\mathcal{N}(0, N)$ and assumed to be independent of the signal $X_i$.

Power constraint

\[ E[X_i^2] \leq P. \]

Definition The information capacity with power constraint $P$ is

\[ C = \max_{E[X_i^2] \leq P} I(X; Y). \]

Theorem The information capacity of Gaussian channel becomes

\[ C = \max_{E[X_i^2] \leq P} I(X; Y) = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right), \]

where the maximum is attained when $X \sim \mathcal{N}(0, P)$.

Definition An $(M, n)$ code for the Gaussian channel with power constraint $P$ consists of the following:

1. An index set $\{1, 2, ..., M\}$.
2. An encoding function $x : \{1, 2, ..., M\} \rightarrow X^n$, yielding codewords $x^n(1), x^n(2), ..., x^n(M)$, satisfying the power constraint $P$; that is for every codeword

\[ \sum_{i=1}^{n} x_i^2(w) \leq nP, w = 1, 2, ..., M. \]

3. A decoding function $g : Y^n \rightarrow \{1, 2, ..., M\}$.

Definition A rate $R$ is said to be achievable with a power constraint $P$ if there exists a sequence of $(2^{nR}, n)$ codes with codewords satisfying the power constraint such that the maximal probability of error $\lambda^{(n)}$ tends to zero. The capacity of the channel is the supremum of the achievable rates.

2 The Coding Theorem for the Gaussian Channel and the Converse

Theorem The capacity of a Gaussian channel with power constraint $P$ and noise variance $N$ is

\[ C = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right) \text{ bits per transmission.} \]

Conversely, the rates $R > C$ are not achievable.

*Based on Cover & Thomas, Chapter 9
3 Bandlimited Channels

Definition A bandlimited continuous-time channel with white noise:

\[ Y(t) = (X(t) + Z(t)) * h(t), \]

where \( X(t) \) is the signal waveform, \( Z(t) \) is the waveform for the white Gaussian noise, and \( h(t) \) is the impulse response of an ideal bandpass filter, which cuts out all frequencies greater than \( W \).

Theorem Suppose that a function \( f(t) \) is bandlimited to \( W \), namely, the spectrum of the function is 0 for all frequencies greater than \( W \). Then the function is completely determined by samples of the function spaced \( 1/2W \) seconds apart.

- Capacity of discrete-time Gaussian channel:
  \[ C = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right) \text{ bits per transmission.} \]

- Capacity per sample:
  \[ C = \frac{1}{2} \log \left( 1 + \frac{P/2W}{N_0/2} \right) = \frac{1}{2} \log \left( 1 + \frac{P}{N_0W} \right) \text{ bits per sample.} \]

- Capacity of a bandlimited continuous-time channel with white noise:
  \[ C = W \log \left( 1 + \frac{P}{N_0W} \right) \text{ bits per second.} \]

- \( W \to \infty \) gives
  \[ C = \frac{P}{N_0} \log_2 e \text{ bits per second.} \]

4 Parallel Gaussian Channels

Definition (Parallel Gaussian channel) A set of Gaussian channels in parallel:

\[ Y_j = X_j + Z_j, \quad j = 1, 2, ..., k, \]

with

\[ Z_j \sim \mathcal{N}(0, N_j), \]

and the noise is assumed to be independent from channel to channel.

Power constraint

\[ E \left[ \sum_{j=1}^{k} X_j^2 \right] \leq P. \]
Theorem The information capacity of the channel $C$ is
\[
C = \max_{\sum_{i=1}^{k} X_i^2 \leq P} I(X_1, X_2, ..., X_k; Y_1, Y_2, ..., Y_k),
\]
where $P_i = E[X_i^2]$, and $\sum P_i = P$. Equality is achieved by
\[
(X_1, X_2, ..., X_k) \sim \mathcal{N} \left( \begin{bmatrix} 0 & 0 & \cdots & 0 \\ P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k \end{bmatrix} \right).
\]

Theorem (Water filling) $P_i$ assignment that maximizes capacity:
\[
P_i = (\nu - N_i)^+,
\]
where $\nu$ is chosen so that
\[
\sum P_i = \sum (\nu - N_i)^+ = P.
\]

5 Channels with Colored Gaussian Noise

Theorem Let $K_Z$ be the covariance matrix of the noise and $K_X$ be the input covariance matrix. The power constraint on the input can be
\[
\frac{1}{n} \sum_i E[X_i^2] = \frac{1}{n} \text{tr}(K_X) \leq P.
\]
We need to maximize $h(Y_1, Y_2, ..., Y_n)$ and it is maximized when the input is normal with a trace (power) constraint on $K_X$.

Theorem Let us decompose $K_Z$ into its diagonal form,
\[
K_Z = Q \Lambda Q^t, \quad \text{where } QQ^t = I,
\]
where $\Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$, and let $A = Q^t K_X Q$. The maximum is attained when
\[
A_{ij} = \begin{cases} 
(\nu - \lambda_i)^+, & i = j, \\
0, & i \neq j,
\end{cases}
\]
where $\nu$ is chosen so that $\sum A_{ii} = \sum (\nu - \lambda_i)^+ = nP$.

6 Gaussian Channels with Feedback

Definition (Gaussian channel with feedback):
\[
Y_i = X_i + Z_i, \quad Z_i \sim \mathcal{N}(0, K_Z^{(n)})
\]
Definition (Sequence of mappings) :

\[ x_i(W, Y_{i-1}), \]

where \( W \in \{1, 2, ..., 2^{nR}\} \) is the input message and \( Y_{i-1} \) is the sequence of past values of the output.

Power constraint:

\[ E \left[ \frac{1}{n} \sum_{i=1}^{n} x_i^2(w, Y_{i-1}) \right] \leq P, \quad w \in \{1, 2, ..., 2^{nR}\}. \]

1. With feedback. The capacity \( C_{n, FB} \) in bits per transmission of the time-varying Gaussian channel with feedback is

\[ C_{n, FB} = \max_{\frac{1}{n} \text{tr}(K_X^{(n)}) \leq P} \frac{1}{2n} \log \frac{|K_X^{(n)} + Z^{(n)}|}{|Z^{(n)}|^n}. \]

2. Without feedback. The capacity \( C_n \) in bits per transmission of the time-varying Gaussian channel without feedback is

\[ C_n = \max_{\frac{1}{n} \text{tr}(K_X^{(n)}) \leq P} \frac{1}{2n} \log \frac{|K_X^{(n)} + K_Z^{(n)}|}{|K_Z^{(n)}|^n}. \]

Without feedback, achieving capacity in the time-varying case reduces to water-filling on the eigenvalues \( \{\lambda_i^{(n)}\} \) of \( K_Z^{(n)} \).

**Theorem** For a Gaussian channel with feedback, the rate \( R_n \) for any sequence of \((2^{nR}, n)\) codes with \( P_e^{(n)} \to 0 \) satisfies

\[ R_n \leq C_{n, FB} + \epsilon_n, \]

with \( \epsilon_n \to 0 \) as \( n \to \infty \).

This upper bound is in fact achievable (not proved here), and so \( C_{n, FB} \) is in fact the capacity of the Gaussian channel with feedback.

**Theorem**

\[ C_{n, FB} \leq C_n + \frac{1}{2} \] bits per transmission.

**Theorem**

\[ C_{n, FB} \leq 2C_n. \]

**Conclusion** Feedback can increase the capacity of a Gaussian channel with memory, but the increase is no more than the minimum of an additive 1/2 bit per transmission or a factor of two.