Ignoring $R_D$, we can draw the following small-signal model:

\[ \text{Rin} = R_B \parallel R_t \]
\[ R_B = R_l \parallel R_2 \]
\[ R_t = \frac{V_t}{I_t} = \frac{V_t}{I_b_1} \quad (\therefore I_b_1 = I_t, \text{ neglecting } R_B) \]

\[ \text{We assume } R_B \text{ to be much larger than } R_\pi \]
\[ \therefore I_b_2 = -\beta I_b_1 \]

\[ I_b_1 = (V_t - V_e) \times \frac{1}{R_\pi} \quad - (1) \]
\[ V_e = \beta^2 I_b_1 + \beta I_b_1 + I_b_1 = \beta(\beta + 1) I_b_1 \times R_e \quad - (2) \]

On substituting $V_e$ in (1) we get

\[ \frac{V_t}{I_t} = R_t = R_\pi + \beta(\beta + 1) R_e \]
Another quick way of finding the input impedance is from the following knowledge:

\[ R_{in} \rightarrow K \rightarrow \Rightarrow R_{in} \rightarrow i_B \rightarrow R_T \rightarrow \beta i_B \rightarrow \Rightarrow R_{in} \rightarrow \]

We know (from previous experience), \( R_{in} \) for the above emitter-follower configuration is

\[ R_{in} = R_T + (\beta+1)R_e \]

We can extend the above idea to the two-stage amplifier. Since there is an extra \( \beta \) multiplication due to the second stage, we can approximately say

\[ R_{it} = R_T + \beta(\beta+1)R_e. \]

Since \( R_e \) is multiplied by \( \beta(\beta+1) \), \( R_{in} \) is going to be high for moderate \( R_e \) and \( \beta \) values.

So, our first assumption of the input impedance is a fair one.
**OUTPUT IMPEDANCE:**

Again, neglecting the output impedance of the BJTs, we can draw the low-frequency small-signal model. In order to make the derivation simpler without loss of accuracy, we find the output impedance with the input port shorted. We also assume $I_{RB}$ is negligible.

![Circuit Diagram]

**Figure**

**KCL at output node**

\[ i_{q} = B^2 \cdot i_{b} + \frac{V_{o} - V_{e}}{R_{C}} \]

\[ = V_{e} \left( \frac{B^2}{r_{\pi}} - \frac{1}{R_{C}} \right) + \frac{V_{o}}{R_{C}} \quad - (1) \]

**KCL at node $V_{e}$**

\[ \frac{V_{o} - V_{e}}{R_{C}} = B \cdot \frac{V_{e}}{r_{\pi}} + \frac{V_{e}}{r_{\pi}} + \frac{V_{e}}{R_{C}} \quad - (2) \]

or \[ \frac{V_{o}}{R_{C}} = V_{e} \times \left\{ \frac{1}{R_{C}} + \frac{1}{R_{C}} + \frac{B+1}{r_{\pi}} \right\} \]
Substituting $v_e$ from eqn 2 in eqn 1

we get

$$i_o = \frac{v_o \times \text{Re}}{(\text{Re} \times \text{f}_\alpha + \text{Re} \times \text{f}_\alpha + (\beta + 1) \text{Re} \times \text{Re})} \times \left(\frac{\text{Re} \times \beta^2 - \text{f}_\alpha}{\text{Re}}\right) + \frac{v_o}{\text{Re}}$$

Assuming $(\beta + 1) \text{Re} \times \text{Re} \gg \text{f}_\alpha \times (\text{Re} + \text{Re})$

& $\beta^2 \text{Re} \gg \text{f}_\alpha$, $i_o$ simplifies to:

$$i_o \approx v_o \frac{(\beta + 1)}{\text{Re}}$$

or

$$\boxed{i_o = \frac{\text{Re}}{(\beta + 1)}}$$

Low output impedance.

An intuitive way of looking at it:

If you have voltage source (say $v_o$) at the output, then $v_e$ is going to change with the divider ratio of $\text{Re}$ & $\text{Re}$. This will inject a current in $v_e$ into the base of Q2 which will further get injected (multiplied by $\beta$) into the output and thus appearing to be a low output impedance.

* So, our assumption of a low output impedance also seems to be a fair assumption for this topology.
Now we can draw the complete Wien-Bridge oscillator with the two-port model of the amplifier.

It can be redrawn in the following manner:

where $R_1 = R + Rout$ and $R_2 = R || Rin$.
AMPLITUDE CONTROL

Without going into the rigorous treatment of amplitude control, we'll look at it qualitatively on how the amplitude stabilizes. Once we choose the gain \((1 + \frac{R_s}{R_c})\) to be greater than 3, it sets the condition for the amplitude of oscillation to grow indefinitely.

But in reality, as the amplitude starts approaching the supply voltage, it will limit itself although the input to the amplifier seems to be increasing. This will look like a 'gain decreasing' activity.
FREQUENCY STABILITY: It is one of the most desirable properties of an oscillator. Once designed to oscillate at a certain frequency, you want minimum deviation from the desired value for temperature drift, component variation, circuit noise, etc.

* * PHASE SENSITIVITY: is one of the figure-of-merits which directly indicates frequency stability. It can be defined as \( \frac{d\omega}{d\theta} = \frac{1}{d\theta/d\omega} \)

* * Phase Sensitivity for Wein-Bridge Oscillator: Since \( \beta(s) \) was the only frequency selective network, we can find the sensitivity of \( \beta(j\omega) \). We have already shown that:

\[ \beta(j\omega) = \frac{R}{3R + j(\omega R^2 - \frac{1}{\omega C})} \]

\[ \phi = \angle \beta(j\omega) = \tan^{-1} \left( \frac{\omega R^2 - \frac{1}{\omega C}}{3R} \right) \]

Since we are interested in the phase @ \( \phi \approx 0^\circ \)

\[ \phi \approx -\frac{(\omega R^2 - \frac{1}{\omega C})}{3R} \]
\[
\frac{d\Phi}{dw} = -\frac{1}{3R} \left( \frac{1}{\omega^2 C} \right)
\]
\[
\left. \frac{d\Phi}{dw} \right|_{\omega=\omega_0} = -\frac{1}{3R} \left( R^2 C + \frac{R^2 C^2}{C} \right) = -\frac{2RC}{3} = -\frac{2}{3\omega_0}
\]

\[
\left( \frac{d\Phi}{dw} \right) = \frac{2}{3\omega_0}
\]

or
\[
\frac{1}{\left( \frac{d\Phi}{dw} \right)} = \frac{\omega_0}{\frac{2}{3}}
\]

* This is usually high for any precision frequency generation.

* Next, we'll look at LC based oscillators which offer better frequency stability.
LC OSCILLATORS:

The Wien-Bridge oscillator which we analyzed in the previous section, is a R-C based oscillator. If you want good frequency stability, R-C based oscillators are not your best choice. We'll look at another family of oscillators which are more suitable for superior frequency stability.

These are based on inductor-capacitor (LC) combination to provide the frequency-selective network. Now, in reality we can never have L & C only. There will be always some loss associated the capacitors or inductors which can be represented either as a series or parallel resistor.

Before we analyze some of the common LC oscillator configuration, let's look at one of the popular L-R-C configuration, the parallel L-R-C configuration or lot of times referred to as L-C tank, R is called the loss of the tank.
PARALLEL RLC TANK CIRCUIT

\[ Y \rightarrow R \quad C \quad L \]

It's easier to analyze the admittance of a parallel combination by finding the admittance of the network.

\[ Y(j\omega) = \frac{1}{R} + j\omega C + \frac{1}{j\omega L} \]

\[ = \frac{1}{R} + j(\omega C - \frac{1}{\omega L}) \]

As seen from the network and as well from admittance equation, the admittance is infinity at \( \omega = \infty \) & \( \omega = 0 \). And, the admittance becomes real for one frequency (say \( \omega_0 \)) when the imaginary portion of the admittance becomes zero. i.e.,

\[ (\omega_0 C - \frac{1}{\omega_0 L}) = 0 \quad \text{or} \quad \omega_0 = \frac{1}{\sqrt{L C}} \]

\( \omega_0 \) is known as the 'resonant' frequency of the tank, where the reactances cancel each other and you get only a real component i.e.,

\[ Y(j\omega_0) = \frac{1}{R}. \]
TheQuality factor \( Q \): 

Aside from the resonant frequency, another important descriptive parameter is the quality factor or simply \('Q'\).

There are many definitions of it, perhaps the most fundamental one is as follows:

\[
Q = \omega \cdot \frac{\text{energy stored}}{\text{average power dissipated}}
\]

Note, that \( Q \) is dimensionless and that \( \omega \) is proportional to the ratio of energy stored to the energy lost, per unit time.

Also note that the notion of \( Q \) applies both to resonant and non-resonant system.

Let's now use the definition of \( Q \) to derive \( \theta \) for the parallel RLC circuit. In order to find the power dissipated we'll need to excite the network with say a current source \( I_{\text{in}} \):

\[ I_{\text{in}} \quad \overset{\text{\( R \)}}{\text{\( \bigcirc \)}} \quad \overset{\text{\( C \)}}{\text{\( \bigcirc \)}} \quad \overset{\text{\( L \)}}{\text{\( \bigcirc \)}} \quad \text{\( \overset{\text{\( V_{\text{out}} \)}}{\text{\( \bigcirc \)}} \)} \]
At resonance \((w_0)\), the voltage across the network is simply \(I_{in} \cdot R\). Now, remember in tank the energy sloshes between the capacitor & the inductor, with a constant sum at resonance. As a consequence, the peak energy stored in either the capacitor or the inductor is equal to the total energy stored in the tank at any instance of time.

Since we happen to know the peak voltage \(I_{pk} \cdot R\) max = \(I_{pk} \cdot R = V_{pk}\)

So we can choose the capacitors peak energy and that'll be the total energy stored in the tank at any point of time.

\[ E_{tot} = \frac{1}{2} C \cdot V_{pk}^2 = \frac{1}{2} C \cdot (I_{pk} \cdot R)^2 \]

Average power dissipated by the resistor (assuming sinusoidal excitation) is

\[ P_{avg} = \frac{1}{2} I_{pk}^2 R. \]

From the defn. of \(Q\)

\[ Q = \frac{w_0 \cdot E_{tot}}{P_{avg}} = \frac{1}{\sqrt{LC}} \cdot \frac{\frac{1}{2} C \cdot (I_{pk} \cdot R)^2}{\frac{1}{2} \cdot I_{pk}^2 R} = \frac{R}{\sqrt{L/C}} \]

\[ Q = \frac{R}{\sqrt{L/C}} \]
The quantity $\sqrt{L/C}$ has the dimension of resistance and is often called the characteristic impedance of the network, which is the magnitude of the reactances of the inductor or capacitance at resonance.

**Phase response of $Y(\omega)$:**

$$Y(\omega) = \frac{1}{R} + j \cdot (\omega \cdot C - \frac{1}{\omega \cdot L})$$

$$\angle Y(\omega) = \tan^{-1} \left( \frac{R \cdot \omega \cdot C}{R \cdot \omega \cdot L} \right)$$

at $\omega = 0$ \hspace{1cm} $\angle Y(\omega) = -90^\circ$ \hspace{1cm} $(-\pi/2)$

$\omega = \infty$ \hspace{1cm} $\angle Y(\omega) = +90^\circ$ \hspace{1cm} $(\pi/2)$

at $\omega = \omega_0$ \hspace{1cm} $\angle Y(\omega) = 0^\circ$. 
Bandwidth is another important measure of a resonant system. It defines the magnitude and phase behaviour around the resonant frequency $\omega_0$. As shown on the graph, the bandwidth is the span of frequency around $\omega_0$ (\(\Delta\omega\)) for which the magnitude varies by 3dB.

The relationship between bandwidth and $Q$ can be shown to be:

$$BW = \frac{\omega_0}{Q}$$
PHASE SENSITIVITY:

For a parallel R-L-C network we know

\[ Q = \frac{R}{\sqrt{R^2 + \omega_0^2 C}} \quad \text{and} \quad \omega_0 = \frac{1}{\sqrt{LC}} \quad (1) \]

we can derive the following expression,

\[ \frac{Q}{\omega_0} = R \cdot C \quad \text{and} \quad Q \cdot \omega_0 = \frac{R}{L} \quad (2) \]

The phase angle of the admittance of a parallel R-L-C network is

\[ \phi(w) = \phi = \tan^{-1}\left\{ \frac{R \cdot (\omega C - \frac{1}{\omega L})}{1} \right\} \]

For phase sensitivity we're interested around \( \omega_0 \) where the phase angle \( = \) zero, that implies

\[ \phi = \tan^{-1}\left\{ \frac{R \cdot (\omega_0 C - \frac{1}{\omega_0 L})}{1} \right\} = R \cdot \omega_0 C - \frac{R}{L} \]

Using the expressions from \( (2) \) we can write

\[ \phi = \frac{Q \cdot \omega}{\omega_0} - \frac{Q \cdot \omega_0}{\omega} = Q \cdot \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \]

Phase sensitivity \( = \frac{d\phi}{dw} \big|_{w=\omega_0} \)
\[ \Phi', \frac{d\Phi}{dw} \bigg|_{w=\omega_0} = \frac{\Phi}{\omega_0} + \frac{\Phi\omega_0}{\omega^2} \bigg|_{w=\omega_0} = \frac{2\Phi}{\omega_0} \]

Phase sensitivity: \[ \frac{d\Phi}{dw} \bigg|_{w=\omega_0} = \frac{2\Phi}{\omega_0} \]

It can be easily shown that the phase sensitivity of the Impedance is \[ \frac{-2\Phi}{\omega_0} \].