Extra credit #1

EE 194: Advanced VLSI
Spring 2018

This extra-credit problem will be due the same time as the regular homework, as indicated on the class calendar. You can either write up your answers and turn this in via *provide*, or just stop by during office hours and explain your solutions in person (without needing to write them down).

Like most(all) of the extra-credit problems, the answers to these questions require material that is not covered in the class, and usually is not in the textbook either.

We mentioned in class that, most of the time, the energy expended in a series resistor is the same as the energy stored or discharged in a capacitor. This is our *energy hypothesis*. Now let's prove when it does (and doesn't) work.

1. (easy) Consider a capacitor *C* initially charged to a voltage *V*. Discharge it by connecting a resistor *R* across its two terminals. Prove that the energy lost by the capacitor all becomes heat energy created by the resistor. Prove that this is true at any point during the discharge process, even before the capacitor is fully discharged. Prove that this energy is independent of *R*.

   We could solve this by using the differential equation for a capacitor, $i = C\frac{dV_C}{dt}$ as well as KCL and KVL, and solving for the current as a function of time. However, as with most physics problems, it's substantially easier to use energy methods. The answer is then almost trivial. At t=0, there is energy stored in the capacitor (in the form of an electric field). As the capacitor discharges to some voltage, there is of course less energy stored in the capacitor. A resistor does not store energy. Thus any energy no longer stored in the capacitor can be in only one place – converted to heat. Note that the value of the resistor did not even enter into the discussion; it is simple conservation of energy.

2. (medium) Now charge *C* via a series circuit: a voltage source *V* through a resistor *R* and then *C*. Prove that when we fully charge *C* from 0 volts to *V* volts, the hypothesis is true. Prove that this holds even if *R* changes while the capacitor is charging (which is good, since an MOS transistor in its active mode can be modeled as a variable resistor). Hint: this problem does take a small bit of calculus, but does not require solving any differential equations.

   This too can be done by solving the usual differential equations. However, it is substantially more complex. Note that the usual result that $V_C = V_s\left(1 - e^{-t/\tau}\right)$ only works for a constant R, which is not the case here; the differential equations are more complex than that.

Instead, we will again use energy methods. Let the current at any time be i. Let the voltages be V_C for the capacitor and V_S for the constant voltage source.

The power supplied by our voltage source V_S at any given time is simply iV_S; therefore the total energy is $E_S = V_S \int idt$. But the voltage-source current is in series with the capacitor current and they are thus equal, so in fact (by the definition of a capacitor), $i = C\frac{dV_C}{dt}$. Then the energy expended by the voltage source is $E_S = V_S \int idt = V_S \int \left(C\frac{dV_C}{dt}\right)dt = CV_S \int dV_C = CV_S^2$. When the charging is done, the amount $.5CV_S^2$ of this will have gone into the capacitor's electric field, leaving the remaining $.5CV_S^2$ to have been expended as heat energy.

Here's an alternate derivation that's a bit harder. The energy stored in the capacitor is $E_C=.5CV_C^2$. But the voltage across a capacitor is $V_C=Q_C/C$, and by the definition of current we must have $Q_C = \int_{t=0}^{t_f} idt$, and thus the capacitor energy is $E_C = .5CV_C^2 = .5Q_CV_C = .5V_C \int idt$.

Next, consider the energy supplied by our voltage source V_S. At any given time, the power is simply iV_S; therefore the total energy is $E_S = V_S \int idt$. But by energy considerations, any energy supplied by the source can only go to two places: it can be stored in the capacitor, or dissipated as heat by the resistor (an ideal capacitor stores energy, but does not dissipate energy). Thus the energy dissipated by the resistor must be $E_R = E_S - E_c = V_S \int idt - .5V_C \int idt$.

We're almost there. When the capacitor is fully charged, then $V_C=V_S$, which implies that $E_R = V_C \int idt - .5V_C \int idt = .5V_C \int idt$. But this is just the expression we've derived above for E_C – thus $E_R=E_C$. As with problem #1, the value of the resistor has not entered into anything we've done. This is as expected: the resistor controls the rate of the process, but does not alter the final equilibrium result.

3. (easy, given #2) Prove that if we only partially charge C (i.e., not all the way to V volts), then the hypothesis is false. Does the charging process waste more energy near the beginning or near the end? There is a style of circuit design called *adiabatic logic* that takes advantage of this.

From above, $E_S = V_S \int idt = V_S \int \left(C\frac{dV_C}{dt}\right)dt = CV_SV_C$. The energy in the capacitor is $.5CV_C^2$, leaving $CV_SV_C - .5CV_C^2 = CV_C(V_S - .5V_C)$ for the resistor. This is only equal to $.5CV_C^2$ when $V_C=V_S$; until then, it is larger. So the charging process is at its most efficient when the two voltages are close together (at the end of the charging), and at its least efficient when V_S far exceeds V_C (at the beginning).

The alternate proof could also be used. We have $E_R = V_S \int idt - .5V_C \int idt$. We can rephrase this as $E_R = \left(\frac{V_S}{.5V_C}\right).5V_C \int idt - .5V_C \int idt = E_C\left(\frac{V_S}{.5V_C} - 1\right)$. Clearly, this is only equal to E_C when $V_S=V_C$. Certainly V_C cannot exceed V_S, but clearly the process is at its most efficient when the two voltages are close together (at the end of the charging), and at its least efficient when V_S far exceeds V_C (at the beginning).

In adiabatic logic, the power supply is raised to its final value slowly, with the result that the resistor burns almost no power. In the language of thermodynamics, we would say that this approaches a reversible process, with little increase in entropy and hence little energy lost as heat.

4. (easy) Now replace the voltage source *V* with a constant-current source *I*. Prove that, for this circuit, the hypothesis is no longer true when charging the capacitor.
   Let the current source have value I_S. Then the resistor power is a constant I_S^2R, and the resistor energy is $I_S^2R\Delta t$. In the capacitor, we have $Q_C = I_S\Delta t$ and the energy stored in the capacitor is $E_C = .5CV_C^2 = .5C\left(\frac{Q_C}{C}\right)^2 = .5C\left(\frac{I_S\Delta t}{C}\right)^2 = .5\frac{I_S^2\Delta t^2}{C}$. Clearly, $I_S^2R\Delta t$ does not in general equal $.5\frac{I_S^2\Delta t^2}{C}$ for all values of *R*, *C* and Δt.

5. (might take some thought) Given that an MOS transistor in saturation is often modeled as a constant-current source, and that we just showed that our energy hypothesis does *not* work for a constant-current source, why do we often claim that our hypothesis works well in integrated circuits?
   This is an interesting question. It is true that an MOS device in saturation is reasonably modeled by a constant-current source at times. However, it is not fair to say that this is identical to the constant-current source we use in circuit analysis. Most importantly, whereas the true constant-current source expends a power at any time of just its current times the instantaneous voltage across it, the MOS device does not obey such a simple rule. With regards to the power it produces, it is probably better modeled as a constant-voltage source in series with a time-varying resistor of just the correct value as to provide the desired "constant" current.