Topic 4: Multivariate random variables

- Joint, marginal, and conditional pmf
- Joint, marginal, and conditional pdf and cdf
- Independence
- Expectation, covariance, correlation
- Conditional expectation
- Two jointly Gaussian random variables

Multiple random variables

- In many problems, we are interested in more than one random variables representing different quantities of interest from the same experiment and the same sample space.
  - Examples: The traffic loads at different routers in a network, the received quality at different HDTVs, the shuttle arrival time at different stations.
- These random variables can be represented by a random vector $\mathbf{X}$ that assign a vector of real number to each outcome s in the sample space Ω.

$$\mathbf{X} = (X_1, X_2, \dots, X_n)$$

- It prompts us to investigate the mutual coupling among these random variables.
  - We will study 2 random variables first, before generalizing to a vector of n elements.

Joint cdf

- The *joint cdf* of two random variables X and Y specifies the probability of the event $\{X \leq x\} \cap \{Y \leq y\}$

$$F_{X,Y}(x,y) = P[X \leq x, Y \leq y]$$

  The joint cdf is defined for all pairs of random variables.

- Properties of the joint cdf:
  - Non-negativity:
$$F_{X,Y}(x,y) \geq 0$$
  - Non-decreasing:
$$\begin{aligned} &\text{If} \quad && x_1 \leq x_2 \text{ and } y_1 \leq y_2, \\ &\text{then} \quad && F_{X,Y}(x_1,y_1) \leq F_{X,Y}(x_2,y_2) \end{aligned}$$
  - Boundedness:
$$\begin{aligned} \lim_{x,y\to\infty} F_{X,Y}(x,y) &= 1 \\ \lim_{x\to-\infty} F_{X,Y}(x,y) &= \lim_{y\to-\infty} F_{X,Y}(x,y) \;=\; 0 \end{aligned}$$

  - Marginal cdf's: which are the individual cdf's
$$F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y)$$
    The marginal cdf's can be obtained from the joint cdf, but usually not the reverse. In general, knowledge of all marginal cdf's is *insufficient* to specify the joint cdf.
  - Rectangle formula:
$$\begin{aligned} & P[a < X \leq b, c < Y \leq d] \\ = \;& F_{X,Y}(b,d) - F_{X,Y}(b,c) - F_{X,Y}(a,d) + F_{X,Y}(a,c) \end{aligned}$$

Discrete random variables – Joint pmf

- Consider two discrete random variables X and Y.
- They are completely specified by their *joint pmf*, which specifies the probability of the event $\{X = x, Y = y\}$

$$p_{X,Y}(x_k, y_j) = P(X = x_k, Y = y_j)$$

- The probability of any event A is given as

$$P((X,Y) \in A) = \sum_{(x_k, y_j) \in A} p_{X,Y}(x_k, y_j)$$

- By the axioms of probability

$$p_{X,Y}(x_k, y_j) \geq 0$$

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} p_{X,Y}(x_k, y_j) = 1$$

Marginal and conditional pmf's

- From the joint pmf $P_{X,Y}(x, y)$, we can calculate the individual pmf $p_X(x)$ and $p_Y(y)$, which are now referred to as the *marginal pmf*

$$p_X(x_i) = \sum_{j=1}^{\infty} p_{X,Y}(x_i, y_j)$$

Similarly $p_Y(y_k) = \sum_{i=1}^{\infty} p_{X,Y}(x_i, y_k)$.

  - The marginal pmf is an one-dimensional pmf.
  - In general, knowledge of all marginal pmf's is *insufficient* to specify the joint pmf.

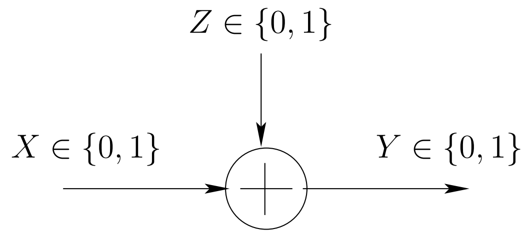
- Sometimes we know one of the two random variables, and we are interested in the probability of the other one. This is captured in the *conditional pmf*

$$p_{X|Y}(X = x_i | Y = y_k) = \frac{p_{X,Y}(x_i, y_k)}{p_Y(y_k)}$$

provided $p_Y(y_k) \neq 0$. Otherwise define $p_{X|Y}(x|y_k) = 0$.

Example: The binary symmetric channel

Consider the following binary communication channel



The bit sent is	$X \sim \text{Bern}(p), \quad 0 \le p \le 1$
The noise is	$Z \sim \text{Bern}(\epsilon), \quad 0 \le \epsilon \le 0.5$
The bit received is	$Y = (X + Z) \mod 2 = X \oplus Z$
	where X and Z are independent.

Find the following probabilities:

1. $p_Y(y)$;
2. $p_{X|Y}(x|y)$;
3. $P[X \neq Y]$, the probability of error.

Jointly continuous random variables – Joint, marginal, and condition pdf

- Two random variables X and Y are *jointly continuous* if the probability of any event involving (X, Y) can be expressed as an integral of a probability density function

$$P[(X,Y) \in A] = \int\int_A f_{X,Y}(x,y)\, dx\, dy$$

  - $f_{X,Y}(x,y)$ is called the *joint probability density function*
  - It is possible to have two continuous random variables that are not jointly continuous.

- For jointly continuous random variables

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\, dv\, du$$

$$f_{X,Y}(u,v) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

- Since $F_{X,Y}(x,y)$ is non-decreasing, the joint density is always non-negative

$$f_{X,Y}(x,y) \geq 0$$

But $f_{X,Y}(x,y)$ is NOT a probability measure (it can be > 1).

- By the axioms of probability,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

- The *marginal pdf* can be obtained from the joint pdf as

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

- The *conditional pdf* is given as

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- Example: Consider two jointly Gaussian random variables with the joint pdf

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\}$$

  - The marginal pdf of X is Gaussian

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

  - The conditional pdf of X given Y is also Gaussian

$$f_X(x|y) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right\}$$

Independence

- Independence between 2 random variables X and Y means

$$P(X \in B, Y \in C) = P(X \in B)\, P(Y \in C)$$

- Two random variables X and Y are *independent* if and only if

$$F_{X,Y}(x, y) = F_X(x) F_Y(y)$$

- Equivalently, in terms of the joint and conditional densities
  - For discrete r.v.'s:

$$\begin{aligned} p_{X,Y}(x_i, y_k) &= p_X(x_i) p_Y(y_k) \\ p_{X|Y}(x_i|y_k) &= p_X(x_i) \end{aligned}$$

  - For jointly continuous r.v.'s:

$$\begin{aligned} f_{X,Y}(x, y) &= f_X(x) f_Y(y) \\ f_{X|Y}(x|y) &= f_X(x) \end{aligned}$$

- X and Y independent implies

$$E[XY] = E[X]E[Y]$$

The reverse is *not* always true. That is, $E[XY] = E[X]E[Y]$ does not necessarily imply X and Y are independent.

- Example: Consider discrete RVs X and Y such that

$$P[X = \pm 1] = P[X = \pm 2] = \frac{1}{4} \quad \text{and } Y = X^2$$

Then $E[X] = 0$ and $E[XY] = 0$, but X and Y are not independent. Find $p_Y(1)$, $p_Y(4)$, $p_{XY}(1, 4)$.

Expectation, covariance, correlation

- Let $g(x,y)$ be a function of two random variables X and Y. The expectation of $g(X,Y)$ is given by

$$E[g(X,Y)] = \begin{cases} \sum_k \sum_i g(x_i, y_k) p_{X,Y}(x_i, y_k) & \text{discrete } X, Y \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy & \text{jointly continuous } X, Y \end{cases}$$

- The *covariance* between 2 random variables is defined as

$$\begin{aligned} \text{cov}(X,Y) &= E\big[X - E[X]\big]\big[Y - E[Y]\big] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

  – If $E[X] = 0$ or $E[Y] = 0$ then $\text{cov}(X,Y) = E[XY]$.

  – Covariance is analogous to the variance of a single random variable: $\text{cov}(X,X) = \text{var}(X)$.

- $E[XY]$ is the *correlation* between the two random variables.

  – By the Cauchy-Schwatz inequality

$$|E[XY]| \le \sqrt{E[X^2]E[Y^2]}$$

- X and Y are said to be (note the difference in terminology here)

  – *orthogonal* if $E[XY] = 0$

  – *uncorrelated* if $\text{cov}(X,Y) = 0$

- The *correlation coefficient* is defined as

$$\rho_{X,Y} = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y}$$

  where $\sigma_X = \sqrt{\text{var}(X)}$ and $\sigma_Y = \sqrt{\text{var}(Y)}$ are the standard deviations.

  Properties:

  – Bounded: $-1 \le \rho_{X,Y} \le 1$

  – If $\rho_{X,Y} = 0$, X and Y are uncorrelated

  – If X and Y are independent, then $\rho_{X,Y} = 0$.
  Independence implies uncorrelation, but not the reverse.

- For Gaussian random variables X and Y, however, if they are uncorrelated, then they are also independent.

Conditional expectation

- The conditional expectation of X given Y is defined as
  - For discrete r.v.'s:

$$E\big[X|Y = y_k\big] = \sum_i x_i \, p_{X|Y}(x_i|y_k)$$

  - For jointly continuous r.v.'s:

$$E\big[X|Y = y\big] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) dx$$

- Similarly for the conditional expectation of a function of X, given Y

$$E\left[g(X)|Y = y\right] = \begin{cases} \sum_i g(x_i) p_{X|Y}(x_i|y) & \text{for discrete } X, Y \\ \int_{-\infty}^{\infty} g(x) \, f_{X|Y}(x|y) dx & \text{for jointly continuous } X, Y \end{cases}$$

- The law of conditional expectation

$$E[g(X)] = E_Y\left[E[g(X)|Y]\right]$$

For discrete r.v.'s

$$E[g(X)] = \sum_k E\big[g(X)|y_k\big] p_Y(y_k)$$

For continuous r.v.'s

$$E[g(X)] = \int_{-\infty}^{\infty} E\big[g(X)|y\big] f_Y(y) dy$$

This law is very useful in calculating the expectation.

- Example: Defects in a chip.
  - The total number of defects on a chip is $X \sim \text{Poisson}(\alpha)$.
  - The probability of finding an defect in the memory is p.

  Find the expected number of defects in the memory.

Two jointly Gaussian random variables

The joint pdf of two jointly Gaussian r.v.'s X and Y is

$$f_{X,Y}(x,y) = \frac{\exp\left\{-\frac{1}{2(1-\rho_{XY}^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho_{XY}\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right\}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}}$$

- The pdf is a function only of μ_X, μ_Y, σ_X, σ_Y, and ρ_{XY}.
- If X and Y are jointly Gaussian then they are individually Gaussian; that is $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$.
- If X and Y are *independent* Gaussian random variables, then they are also jointly Gaussian with the above joint pdf ($\rho_{XY} = 0$).
- In general, however, Gaussian random variables are not necessarily jointly Gaussian.
  - Example: Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 = \begin{cases} +1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$ be independent r.v.'s. Let $X_3 = X_1 X_2$ then $X_3 \sim \mathcal{N}(0,1)$, but X_1 and X_3 are not jointly Gaussian.