Topic 5: Functions of multivariate random variables

- Functions of several random variables
- Random vectors
  - Mean and covariance matrix
  - Cross-covariance, cross-correlation
- Jointly Gaussian random variables

Joint distribution and densities

- Consider n random variables $\{X_1, \ldots, X_n\}$.
- The joint distribution is defined as

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = P[X_1 \le x_1, \ldots, X_n \le x_n]$$

  - Discrete r.v.'s: The joint pmf is defined as

$$p_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = P[X_1 = x_1, \ldots, X_n = x_n]$$

  - Jointly continuous r.v.'s: The joint pdf can be obtained from the joint cdf as

$$f_{X_1,\ldots,X_n}(\mathbf{x}) = \frac{\partial^n}{\partial x_1 \ldots \partial x_n} F_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

- The marginal density is obtained by integrating (summing) the joint pdf (pmf) over all other random variables

$$f_{X_1}(x_1) = \int \cdots \int f_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n) dx_2 \ldots dx_n$$

One function of several random variables

- Let Y be a function of several random variables

$$Y = g(X_1, X_2, \ldots, X_n)$$

To find the cdf of Y, first find the event

$$\{Y \leq y\} \equiv \{R_x \mid \mathbf{x} \in R_x \ , g(\mathbf{x}) \leq y\}$$

then establish

$$F_Y(y) = P[\mathbf{X} \in R_x] = \int \cdots \int_{\mathbf{x} \in R_x} f_X(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

Sum of 2 random variables

- Let X and Y be two random variables and define

$$Z = X + Y.$$

Since $P[Z \leq z] = P[X + Y \leq z]$, the cdf of Z can be expressed as

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x, y) dx dy$$

Thus the pdf of Z is

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} f_{X,Y}(x, z - x) dx$$

- If X and Y are *independent* then the pdf of Z is the *convolution* of the two pdf's

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx$$

- Example: Sum of two (correlated) Gaussian random variables is a Gaussian r.v.

Linear transformation of random vectors

- Let the random vector $\mathbf{Y}$ be a linear transformation of $\mathbf{X}$

$$\mathbf{Y} = \mathbf{A}\mathbf{X}$$

Assume that $\mathbf{A}$ is invertible, then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y}$, and the pdf of $\mathbf{Y}$ is

$$f_Y(\mathbf{y}) = f_X\left(\mathbf{A}^{-1}\mathbf{y}\right) / \det(\mathbf{A})$$

- Example: Linear transformation of 2 jointly Gaussian RVs X and Y

$$\begin{bmatrix} V \\ W \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

where

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\}.$$

Show that V and W are independent, zero-mean Gaussian RVs with variance $1+\rho$ and $1-\rho$.

Transformation of multiple random variables

- Consider multiple functions of multiple jointly continuous random variables X_i as

$$Y_k = g_k(X_1, X_2, \ldots, X_n)\ , \quad k = 1, \ldots, n$$

Assume that the inverse functions exist such that

$$X_i = h_i(Y_1, Y_2, \ldots, Y_n)\ , \quad i = 1, \ldots, n$$

or in the vector form, $X = H(Y)$. Consider the case that these functions are continuous and has continuous partial derivatives. Let

$$dH = \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}$$

then the joint pdf of Y_k is obtained as

$$f_Y(y) = |\det(dH)|\, f_X\left(H(y)\right)$$

where $\det(dH)$ is the Jacobian of H.

- Example: Transformation from the Catersian to polar coordinate.

  Let $X, Y \sim \mathcal{N}(0,1)$ be independent. Find the joint pdf of V and W as

$$\begin{aligned} V &= (X^2+Y^2)^{1/2} \\ W &= \angle(X,Y)\ , \quad W \in [0, 2\pi] \end{aligned}$$

  Inverse transformation: $x = v\cos w$ and $y = v \sin w$. The Jacobian is

$$\mathscr{J} = \begin{vmatrix} \cos w & -v\sin w \\ \sin w & v\cos w \end{vmatrix} = v.$$

  Since $f_{XY}(x,y) = \frac{1}{2\pi}\exp\{-(x^2+y^2)/2\}$, we have

$$f_{V,W}(v,w) = \frac{1}{2\pi} v e^{-v^2/2}\ , \quad v \geq 0, \quad 0 \leq w < 2\pi.$$

  From this, we can calculate the pdf of V as a *Rayleigh* density

$$f_V(v) = v e^{-v^2/2}\ , \quad v \geq 0.$$

  The angle W is uniform: $f_W(w) = \frac{1}{2\pi}$, $w \in [0, 2\pi]$.

  $\Rightarrow$ The radius V and the angle W are independent!

Random vectors, mean and covariance matrices

- Consider a random column vector $\mathbf{X} = [X_1, \ldots, X_n]^T$, X_i are RVs.
- The vector mean is $\mathbf{m}_X = E[\mathbf{X}]$ with elements $\bar{X}_i = E[X_i]$, $i = 1, \ldots, n$
- The *covariance matrix* of a vector $\mathbf{X}$ is defined as

$$\Sigma_X = E\left[(\mathbf{X}-\mathbf{m}_X)(\mathbf{X}-\mathbf{m}_X)^T\right]$$

  which has the element at the position (i,j) as

$$\Sigma_X(i,j) = E\left[(X_i - m_{X_i})(X_j - m_{X_j})\right]$$

- Properties of the covariance matrix
  a) Σ_X is symmetric
  b) The diagonal values are $\Sigma_X(i,i) = \text{var}(X_i)$
  c) Σ_X is non-negative semidefinite, that is

$$\mathbf{a}^T \Sigma_x \mathbf{a} \geq 0 \quad \text{for any real vector } \mathbf{a}$$

  Equivalently, the eigenvalues of Σ_X are non-negative.

- The *correlation matrix* is defined as $\mathbf{R}_X = E[\mathbf{X}\mathbf{X}^T]$. Note that $\Sigma_X = \mathbf{R}_X - \mathbf{m}_X\mathbf{m}_X^T$.

Cross-covariance and cross-correlation matrices

- The *cross-covariance matrix* between two random vectors $\mathbf{X}$ and $\mathbf{Y}$ is

$$\Sigma_{XY} = E\left[(\mathbf{X} - \mathbf{m}_X)(\mathbf{Y} - \mathbf{m}_Y)^T\right]$$

  a) Σ_{XY} is not necessarily symmetric.

  b) $\Sigma_{XY} = \Sigma_{YX}^T$ (the order of $\mathbf{X}$ and $\mathbf{Y}$ matters).

  c) If $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated, then $\Sigma_{XY} = \Sigma_{YX} = \mathbf{0}$.

  d) If we stack two vectors as $\mathbf{Z} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}$ then the covariance matrix of $\mathbf{Z}$ is given by

$$\Sigma_Z = \begin{bmatrix} \Sigma_X & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_Y \end{bmatrix}$$

  If $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated, then Σ_Z is block-diagonal.

- The *cross-correlation matrix* between $\mathbf{X}$ and $\mathbf{Y}$ is

$$R_{XY} = E\left[\mathbf{X}\mathbf{Y}^T\right] = \Sigma_{XY} + \mathbf{m}_X \mathbf{m}_Y^T$$

Jointly Gaussian random variables

- Consider a Gaussian vector $\mathbf{X} = [X_1, \ldots, X_n]^T$ in which X_i are jointly Gaussian with
  - Mean $\mathbf{m}_X = E[\mathbf{X}]$
  - Covariance

$$\Sigma_X = E\left[(\mathbf{X} - \mathbf{m}_X)(\mathbf{X} - \mathbf{m}_X)^T\right]$$

- The pdf of $\mathbf{X}$ is

$$f_X(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} \det(\Sigma_X)^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \mathbf{m}_X)^T \Sigma_X^{-1} (\mathbf{x} - \mathbf{m}_X)\right\}$$

- Linear transformation of a Gaussian vector

$$\mathbf{Y} = \mathbf{A}\mathbf{X}$$

  is a Gaussian vector with mean and covariance as

$$\begin{aligned} \mathbf{m}_Y &= \mathbf{A}\mathbf{m}_X \\ \Sigma_Y &= \mathbf{A}\Sigma_X\mathbf{A}^T \end{aligned}$$