Scaling Laws of Single-Hop Cognitive Networks

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***Abstract*—We consider a *cognitive network* consisting of n cognitive users uniformly distributed with constant density among primary users. Each user has a single transmitter and a single receiver, and the primary and cognitive users transmit concurrently. The cognitive users use single-hop transmission in two scenarios: (i) with constant transmit power, and (ii) with transmit power scaled according to the distance to a designated primary transmitter. We show that, in both cases, the cognitive users can achieve a throughput scaled *linearly* with the number of users n. The first scenario requires the cognitive users to have the transmitter-receiver (Tx-Rx) distance bounded, but it can be arbitrarily large. Then with high probability, any network realization has the throughput scaling linearly with n. The second scenario allows the cognitive Tx-Rx distance to grow with the network at a feasible exponent as a function of the path loss and the power scaling factors. In this case, the average network throughput grows at least linearly with n and at most as $n \log(n)$. These results suggest that single-hop transmission may be a suitable choice for cognitive transmission.**

***Index Terms*—Cognitive network, scaling law, single-hop transmission, random network.**

I. INTRODUCTION

THE scaling law of the capacity of ad hoc wireless networks has been an active area of research. Since the seminal work of Gupta and Kumar [2], the scaling law has been studied for wireless networks with a variety of channel models and communication protocols [3]–[15]. The considered networks usually contain n homogeneous users, each with a transmitter and a receiver, located randomly. The focus is on how the network aggregate throughput (also called the sum rate) scales as the number of users $n \to \infty$. In an extended network, this limit is equivalent to fixing the user density and increasing the network area with n, while in a dense network, it is equivalent to fixing the area and increasing the user density. Scaling results for one type of network can often be transformed to that of the other after appropriate power scaling. For this reason, this paper will focus on extended networks.

The throughput scaling in ad hoc (homogeneous) extended networks depends strongly on the node distribution and the physical-layer processing capability. When no cooperation other than simple decode-and-forward (as in multihop) is allowed, each node treats other signals as interference, the network throughput scales at most as $\sqrt{n}$ [2]. When the nodes are uniformly distributed, a simple nearest-neighbor forwarding scheme achieves a $\sqrt{n/log(n)}$ network throughput [2]. When the nodes are distributed according to a Poisson point process, a backbone-based routing scheme achieves the scaling of $\sqrt{n}$ [13], meeting the upper bound. However, when nodes jointly send and receive messages simultaneously, a much different scaling law emerges. Upper bounds based on the max-flow min-cut bound [4], [5], [9], [11], [15] as well as MIMO techniques [8], [14] have been analyzed for various ranges of the path loss exponent. Specifically, for path loss α between 2 and 3, a hierarchical scheme can achieve an *asymptotic* (as the number of hierarchical levels goes to infinity) throughput growth as $n^{2-\alpha/2}$ [15] (asymptotically linear for $\alpha = 2$). A key step in this scheme is MIMO cooperation among clustered nodes. For path loss greater than 3, the nearest-neighbor multihop scheme is scaling-optimal and achieves a throughput of order $\sqrt{n}$. The development of these scaling laws show that assumptions about the network and the node signal processing capability are crucial to the scaling law.

In this paper, instead of considering a homogeneous ad hoc wireless network, we study a cognitive network consisting of two types of users: primary and cognitive. Recent introduction of secondary spectrum licensing [16] necessitates the study of such networks. In these networks, the cognitive users can opportunistically access the now-exclusive but under-utilized spectrum of the primary users, without degrading their performance. Other scenarios in which two different networks operate concurrently are also applicable.

Of interest in such heterogeneous networks is how much throughput the cognitive users can achieve in the presence of the often higher-prioritized primary users? Specifically, how does this throughput scale with the number of cognitive users? We formulate this problem for a network with multiple primary and multiple cognitive users, in which the cognitive users are uniformly distributed with constant density.

Because of the opportunistic nature of the cognitive users, we consider a network and communication model different from the previously mentioned ad hoc networks. We assume that each cognitive transmitter communicates with a receiver within a distance D_c (a parameter that can possibly scale with n), using *single-hop* transmission. Single hop communication appears suitable for cognitive devices which are mostly short-range. Our results, however, are not limited to short-range communication. There can be other cognitive devices (transmitters and receivers) in between a cognitive Tx-Rx pair. This is different from the local scenarios of ad hoc networks, in which every node is talking to its nearest neighbor.

When the cognitive users transmit with constant power, D_c

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is a constant and does not grow with network size, but it can be arbitrarily large. The way to choose D_c in this case may be as follows. For a given large network (with a given, large number of users n), we pick D_c as large as possible, which could be on the order of the size of that network. Then for all other n (either smaller or larger), we keep the same D_c. This scheme appears feasible in practice, since while the scaling law studies networks with n going to infinity, in practice, we always deal with a network with finite n. Therefore, by fixing D_c for a large n first, we can make this parameter non-restrictive and applicable for a large number of network sizes.

When the cognitive users can scale their transmit power according to the distance to the primary user (similar to the model in [17]), then D_c may scale with the network size by a feasible exponent. This exponent is a function of the path loss exponent α and the rate at which the cognitive transmit power scales with distance. In this case, D_c can grow at the rate of up to $n^{1-2/\alpha}$.

Furthermore, we assume a protected radius around each receiver (primary or cognitive) to prevent infinite interference. Without cooperation, the cognitive receivers simply treat other users' signals as interference.

Within such a network, for both cases of constant and varying cognitive transmit power, we find that the cognitive users' throughput scales *linearly* in the number of users n. Equivalently, as $n \to \infty$, the per cognitive user capacity remains constant. Specifically, with constant transmit power, the throughput averaged over all network realizations scales linearly with n. Furthermore, any specific realization of the network (with arbitrary cognitive users locations) also has the throughput scaling linearly with high probability. The rate of convergence to linear scaling for any network is of the order $\exp(-\kappa\sqrt{n})$ for some positive constant κ. This linear scaling holds for all path loss exponent equal or greater than 2, which is different from the result in [15], in which the scaling is asymptotic (as the number of hierarchical levels goes to infinity) and is linear only for $\alpha = 2$. With distance-dependent transmit power, the average throughput of the cognitive user scales at least linearly with n and at most as $n \log n$. Our results thus suggest that an initial approach to building a scalable cognitive network may involve limiting cognitive transmissions to a single hop. This scheme appears reasonable for secondary spectrum usage, which is opportunistic in nature.

The impact of cognitive users on the primary user can be captured in the amount of generated interference. The average interference remains bounded irrespective of the number of cognitive users [18]. Such interference bounds can help design an exclusive region for the primary user, which is void of cognitive transmitters, to guarantee the primary user certain performance. Because of space limitation, however, we refer readers interested in this topic to [18].

A. Summary of main results

Let S_n denote the aggregate throughput of n cognitive users. S_n is random because of the random users locations. The main results on the scaling of S_n are as follows.

- With constant cognitive transmit power
  - The average throughput $E[S_n]$ is linear in n
  $$E[S_n] = n\bar{C}_1$$
  where $\bar{C}_1$ is a constant average rate per cognitive user.
  - With high probability, S_n is also linear in n. That is
  $$S_n = n\bar{C}_1 + \Delta$$
  where with high probability, $\frac{|\Delta|}{n}$ approaches 0 as $n \to \infty$. Moreover, the convergence in probability is given by
  $$P_\delta \triangleq \Pr\left[\frac{|\Delta|}{n} \geq \delta\right] = \rho^n e^{-\theta_n}$$
  for $\delta > 0$, where $0 \leq \rho \leq 1$ and $0 \leq \theta_n \leq \kappa\sqrt{n} - \log a$ for some positive a and κ independent of n. Thus the rate of the convergence in probability is $O\left(e^{-\kappa\sqrt{n}}\right)$.
- With distance-dependent cognitive transmit power $P = P_c r^\gamma$, where P_c is a constant, r is the distance to a primary transmitter, and $\gamma < \alpha - 2$ is the power scaling factor with α as the path loss exponent.
  - The average throughput $E[S_n]$ scales at least linearly in n and at most as $n \log n$.
  $$n\bar{C}_1(\gamma) \leq E[S_n] \leq n(\log n)\bar{C}_2(\gamma)$$
  for some constant $\bar{C}_1(\gamma)$ and $\bar{C}_2(\gamma)$ dependent on γ.
  - Cognitive Tx-Rx distance D_c can grow with the network as
  $$D_c \leq K_d r^{\gamma/\alpha} < K_d r^{1-2/\alpha}$$
  where K_d is a constant. D_c can grow with an exponent of up to $1 - 2/\alpha$. For a large path loss exponent α, this growth is almost at the same rate as that of the network.

B. Paper structure

This paper structure is as follows. In Section II, we introduce the network model and formulate the problem. In Section III, we study the throughput scaling of the cognitive users with constant transmit power in the presence of multiple primary users. In Section IV, we investigate the option of allowing the cognitive users to scale the transmit power according to the distance to a designated primary user. Finally, we provide some concluding remarks in Section V.

II. Problem Formulation

Consider a cognitive network with two types of users: primary and cognitive users. Of interest is how the aggregated throughput of the cognitive users scales with network size, given the presence of the primary users. We first discuss the network model and the channel and signal models. We then formulate the cognitive users throughput scaling problem.

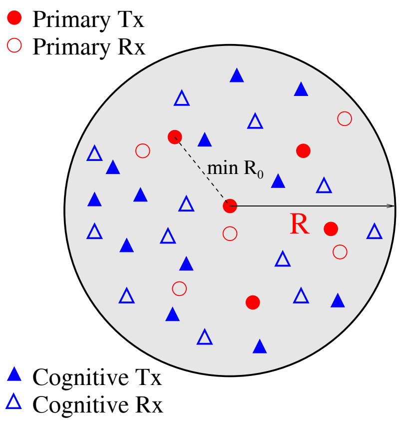

Fig. 1. Cognitive network model with m arbitrarily located primary users and n uniformly located cognitive users.

A. Network model

We consider an extended network with all transmitters and receivers located on a plane. With fixed nodes densities, the network size grows with the number of nodes. As a specific instance, we consider a circular network with radius R. To scale the number of cognitive and primary users, we let R increase. Other network shapes also produce a similar scaling law.

We introduce our network model in Figure 1. Within the network, there are m primary users and n cognitive users. Let Tx_p^i and Rx_p^i denote a primary transmitter and its intended receiver ($i = 1, 2, \cdots, m$), and Tx_c^j and Rx_c^j for the cognitive transmitter and receiver ($j = 1, 2, \cdots, n$).

Around each receiver, either primary or cognitive, we assume a protected circle of radius $\epsilon > 0$ in which no interfering transmitter may lie. (Other receivers, however, can lie within this ϵ circle.) This practical constraint simply ensures that the interfering transmitters and receivers are not located at exactly the same point. It prevents the (aggregated) interference to reach infinity.

Other than the receiver protected regions, the primary transmitters and receivers locations are arbitrary, subject to a minimum distance R_0 between any two primary transmitters. This scenario can correspond to (but is not limited to) a broadcast network, such as the TV or the cellular networks, in which the primary transmitters are base-stations. The cognitive transmitters, on the other hand, are uniformly and randomly distributed with constant density λ. We assume that each cognitive receiver is within a distance D_c from its transmitter. Depending on the transmit power of the cognitive users, D_c may scale with the network size (as analyzed later). Figure 2 provides an example of a such cognitive Tx-Rx layout. We summarize the network notation in Table I.

B. Channel and signal models

Consider a path-loss only model for the wireless channel. Given a distance d between the transmitter and the receiver, the channel h is given as

$$h = \frac{A}{d^{\alpha/2}} \tag{1}$$

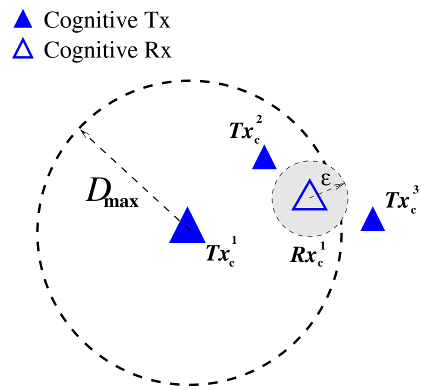

Fig. 2. Cognitive user placement model with maximum Tx-Rx distance D_c and a receiver protected radius ϵ.

TABLE I
NETWORK PARAMETERS.

Network radius	R
Primary user ith transmitter and receiver	Tx_p^i, Rx_p^i
Cognitive user ith transmitter and receiver	Tx_c^i, Rx_c^i
Number of primary users	m
Number of cognitive users	n
Receiver protected radius	ϵ
Minimum primary transmitters distance	R_0
Cognitive user density	λ

where A is a frequency-dependent constant and α is the power path loss. In subsequent analysis, we normalize A to be 1 for simplicity. In this model, d is usually normalized according to a unit length such that effectively $d > 1$ [19]. Assume $\alpha > 2$ which is typical in practical scenarios.

For the signal model, we assume that each user, either primary or cognitive, simply treats other users' signals as noise. Let each primary user transmit with a constant power P_0, and each cognitive user with power P. (Later on we will consider two cases: P is constant, and P is variable with distance). We assume zero-mean Gaussian transmit signals for both types of users. Furthermore, the transmit signals of different users are statistically independent. The receive signals are all corrupted by zero-mean additive white Gaussian noise of power σ^2.

C. The cognitive network throughput

Consider the transmission rate of a cognitive user in the presence of other cognitive users and multiple primary users. Denote I_i^c and I_i^p $(i = 0, \ldots, n)$ as the aggregate interference power to cognitive user i from other cognitive transmitters and from the primary transmitters, respectively. Let h_{ij} denote the channel from cognitive Tx_c^i to cognitive Rx_c^j, and g_{ij} denote the channel from primary Tx_p^i to cognitive Rx_c^j, then these interference powers can be written as

$$I_i^c = \sum_{j \neq i} P|h_{ji}|^2 \; , \quad I_i^p = \sum_{j=1}^{n} P_0 |g_{ji}|^2. \tag{2}$$

Because of the random locations of the cognitive users (transmitters and receivers), the channels gains $|h_{ji}|^2$ and $|g_{ji}|^2$ (as modeled in (1)) are random. Hence both I_i^c and I_i^p are random variables. With Gaussian signaling and transmit power P, the transmission rate of cognitive user i is then

$$C_i = \log\left(1 + \frac{P|h_{ii}^c|^2}{I_i^c + I_i^p + \sigma^2}\right), \quad i = 1 \ldots n. \tag{3}$$

Since the interferences are random, the rate C_i is also random.

Define the aggregate throughput (the sum rate) of the cognitive users as

$$S_n = \sum_{i=1}^{n} C_i. \tag{4}$$

We are interested in how the throughput (4) scales as $n \to \infty$. Since S_n is random, we will study the scaling of its mean first, then analyze the concentration of the throughput around its mean to obtain the scaling of a specific network in probability.

D. Single-primary network with cognitive power scaling

As a special case, we also consider a network with a single primary user at the center. A feasible option is to allow the cognitive transmitters to scale their power according to the distance from the primary user [17]. Specifically, the transmit power P of a cognitive user is now a function of the radius r, at which this cognitive user is located, as

$$P(r) = P_c r^{\gamma} \tag{5}$$

for some constant power P_c and a feasible power exponent γ (which will be analyzed later). The same throughput expressions (3) and (4) hold with the new power defined in (5). We will also examine the throughput scaling of the cognitive users in this case.

III. The Scaling Law of a Cognitive Network with Constant Power

In this section, we study the throughput scaling of the cognitive users with constant transmit power, assuming multiple primary users. We first study the scaling law of the *average* (mean) throughput. We establish upper and lower bounds on the average per-user throughput and show that both bounds scale with the same order, which then becomes the scaling order of the mean throughput itself. We then analyze the concentration of the instantaneous, *random* throughput around its mean and show that with high probability, this instantaneous throughput also scales with the same order, which is linear.

A. Lower bound on the cognitive user average capacity

To derive a lower bound on the average capacity of a cognitive user, we study an upper bound on the interference to a cognitive receiver. This includes the interference from all primary users and all other cognitive users.

1) Interference from the primary users: Recall the assumption that the primary transmitters must be spaced at least a distance R_0 apart (R_0 here is arbitrary and can be taken as the smallest distance between any two primary transmitters, for example). Draw a circle of radius R_0 around each primary transmitter, then the two primary transmitters closest to each other may have these circles touching at one point. We shall

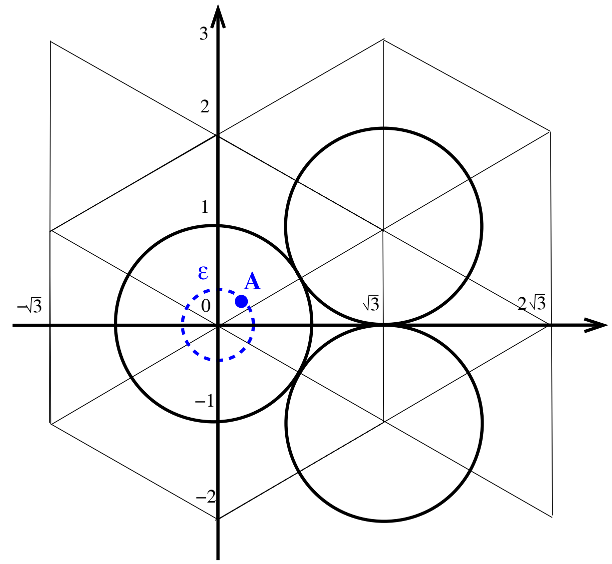

Fig. 3. Worst-case interference when the primary users are located on a hexagon lattice.

examine the densest placement of the primary transmitters to upper bound the interference from them to a cognitive receiver.

Consider primary-transmitter boundary circles with radius R_0. The tightest circle packing is according to the hexagon lattice [20], as shown in Figure 3, in which the three bold circles all have radius R_0 (normalized to 1 unit length in the figure). In the worst case, all primary transmitters are located at the lattice points. Since each cognitive receiver has a protected radius of ϵ, the worst-case cognitive receiver will be on a circle of radius ϵ around a primary transmitter. We are interested in the interference from all the primary transmitters, located on the hexagon lattice, to this cognitive receiver.

To calculate the interference, we superimpose x-y axes on the hexagon topology with the origin at a primary transmitter and with normalized length (1 unit length = R_0), as shown in Figure 3. Consider the circle of radius ϵ around the origin. (For practical reasons, we assume $\epsilon < R_0$.) Let A be an arbitrary point on this circle at an angle θ to the x-axis. The aggregate interference from all primary transmitters to A is given in (6). For any $\theta \in [0, 2\pi]$, each of the summations in (6) is bounded for $\alpha > 2$, as shown in the Appendix. Let $I_{\max}^p$ be the worst-case interference from the primary users,

$$I_{\max}^p = \max_{\theta} I_A \tag{7}$$

then $I_{\max}^p$ is also bounded. Thus the total interference from all primary users to any cognitive receiver is bounded.

2) Worst-case interference from cognitive users: For the worst-case interference, consider a uniform network of n cognitive users, ignoring any protected region around the primary receivers. The worst case interference would then be to a cognitive receiver at the center of the network, which is without loss of generality assumed to be Rx_c^1. From Rx_c^1, draw a circle of radius R that covers all other cognitive transmitters. With constant user density of λ users per unit area, then R^2 grows linearly with n.

To see that this case is indeed the worst interference from cognitive users, consider another cognitive receiver (Rx_c^2) that

$$I_A = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{1}{\left[\left(2\sqrt{3}k - \epsilon\cos\theta\right)^2 + (2m - \epsilon\sin\theta)^2\right]^{\alpha/2}} + \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{1}{\left[\left(\sqrt{3}(2k+1) - \epsilon\cos\theta\right)^2 + (2m+1 - \epsilon\sin\theta)^2\right]^{\alpha/2}}. \quad (6)$$

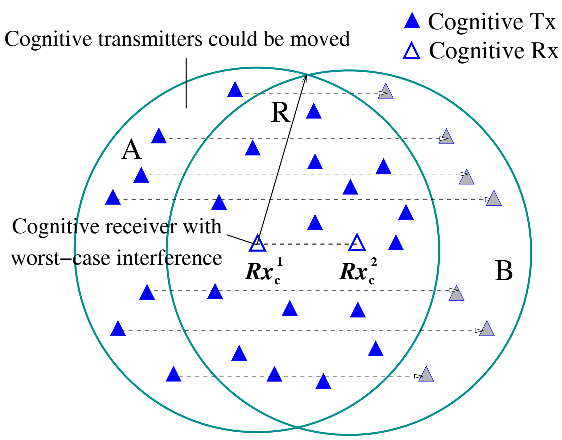

Fig. 4. Worst-case interference to a cognitive receiver.

is not at the center of the network. Again draw a circle of radius R centered at Rx_c^2. Since this receiver is not at the center of the network, the circle will not cover all cognitive transmitters. The interference to Rx_c^2 is then increased by moving all the transmitters from outside this new circle (area A in Figure 4) to inside the circle (area B in Figure 4), resulting in the same interference as to Rx_c^1.

To compute the interference to the worst-case Rx_c^1, consider an interfering cognitive transmitter located randomly within the circle of radius R centered at Rx_c^1. With uniform location distribution, the distance r between this interfering transmitter and Rx_c^1 has the density

$$f_r(r) = \frac{2r}{R^2 - \epsilon^2}, \quad \epsilon \le r \le R, \quad (8)$$

where ϵ again is the protected radius around each cognitive receiver. Therefore the *average interference* from this transmitter to Rx_c^1 is

$$\bar{I}_1 = \int_\epsilon^R \frac{2rP}{(R^2-\epsilon^2)r^\alpha} dr = \frac{2P}{(R^2-\epsilon^2)(\alpha-2)} \left(\frac{1}{\epsilon^{\alpha-2}} - \frac{1}{R^{\alpha-2}}\right). \quad (9)$$

The average aggregate interference from all other cognitive transmitters to Rx_c^1 then becomes

$$\bar{I}_n = n\bar{I}_1 .$$

But $\lambda\pi(R^2 - \epsilon^2) = n$, thus

$$\bar{I}_n = \frac{2\pi\lambda P}{(\alpha-2)} \left(\frac{1}{\epsilon^{\alpha-2}} - \frac{1}{R^{\alpha-2}}\right). \quad (10)$$

As $n \to \infty$, provided that $\alpha > 2$, this average interference to Rx_c^1 approaches a constant as

$$\bar{I}_n \overset{n\to\infty}{\longrightarrow} \frac{2\pi\lambda P}{(\alpha-2)\epsilon^{\alpha-2}} \triangleq \bar{I}_\infty. \quad (11)$$

Since (10) is the worst-case interference, for any cognitive receiver, the average interference from other cognitive users in (3) is upper-bounded by $\bar{I}_n$ as

$$E[I_i^c] \le \bar{I}_n \le \bar{I}_\infty . \quad (12)$$

3) Lower bound on the cognitive user average capacity: Now consider the transmission rate of the ith cognitive user given in (3). Since the distance between a cognitive transmitter and its intended receiver is bounded by D_c, we have $|h_{ii}|^2 \ge 1/D_c^\alpha$. Thus there exists a minimum cognitive received power denoted as

$$P_{r,\min} = \frac{P}{D_c^\alpha}. \quad (13)$$

Given that the interference from the primary users is bounded by $I_{\max}^p$ in (7), then from (3),

$$C_i \ge \log\left(1 + \frac{P_{r,\min}}{\sigma^2 + I_{\max}^p + I_i^c}\right). \quad (14)$$

Noting that $\log(1+a/x)$ is convex in x for $a > 0$, by Jensen's inequality, we have

$$E\log\left(1 + \frac{a}{X}\right) \ge \log\left(1 + \frac{a}{EX}\right).$$

Thus the *average* capacity of each cognitive user satisfies

$$E[C_i] \ge \log\left(1 + \frac{P_{r,\min}}{\sigma^2 + I_{\max}^p + E[I_i^c]}\right) = \log\left(1 + \frac{P_{r,\min}}{\sigma^2 + I_{\max}^p + \bar{I}_n}\right). \quad (15)$$

As $n \to \infty$, the lower bound approaches a constant as

$$E[C_i] \ge \log\left(1 + \frac{P_{r,\min}}{\sigma^2 + I_{\max}^p + I_\infty}\right) \quad (16)$$

where I_∞ is defined in (11). Thus the average per-user capacity of a cognitive network remains at least a constant as the number of users increases. This implies the *average* throughput of the cognitive users scales at least linearly with the number of users.

B. Upper bound on the cognitive network average throughput

A straightforward upper-bound can be obtained by removing the interference from all other cognitive users. Since the number of primary users m is independent of n, the interference from these primary users can be considered as a fixed amount as n grows (this interference can also be lowered bounded by zero). Because of constant transmit power, the capacity of a single cognitive user under noise and interference from the primary users is bounded by a constant. Thus the throughput of all cognitive users grows at most linearly with the number of users.

C. Linear scaling law of the cognitive network average throughput

From the above lower and upper bounds, we conclude that the *average* throughput of the cognitive users grows linearly in the number of users. Specifically, the throughput as defined in (4) satisfies

$$E[S_n] = n\bar{C}_1 \tag{17}$$

where $\bar{C}_1 = E[C_1]$ is the expected rate of a cognitive user, which is a constant larger than or equal to the lower bound in (16). In other words, the *average* per-user transmission rate stays constant as the number of users increases.

D. Concentration of the network throughput around its mean

Given that the *average* cognitive users' throughput scales linearly with the number of users, the concentration of the throughput around its mean is also of interest. This concentration provides the probability that the throughput of a specific network (with a specific realization of the cognitive user locations) scales at the same rate as the mean throughput. Suppose this specific throughput can be written as

$$S_n = E[S_n] + \Delta = n\bar{C}_1 + \Delta$$

for some real Δ. Then we need to show that with high probability, $\frac{|\Delta|}{n}$ approaches 0 as $n \to \infty$.

Specifically, for a $\delta > 0$, we examine

$$P_\delta \triangleq \Pr\left[\frac{1}{n}|S_n - E[S_n]| \geq \delta\right],$$

where $S_n = \sum C_i$ as given in (4). The C_i are i.i.d. random variables with finite mean and finite variance, because of the finite cognitive transmit power.

Based on Chernoff's theorem (Theorem 9.3, [21]), if we let $X_i = C_i - \bar{C}_1 - \delta$ and let $M(t) = E\left[e^{tX_1}\right]$ be their common moment generating function, then

$$\lim_{n\to\infty} \frac{1}{n} \log P_\delta = \log \rho \tag{18}$$

where $\rho = \inf_t M(t)$. Furthermore, $0 \leq \rho \leq 1$, and that the rate of convergence is $O(n^{-1/2})$. The arguments for showing $0 \leq \rho \leq 1$ can be deduced from [21] as follows. Noting that $E[X_i] < 0$, then the first derivative $M'(0) < 0$. For sufficiently small δ, $\Pr[X_i > 0] > 0$, thus $M(t) \to \infty$ as $t \to \infty$. Now since $M(t)$ is convex, decreasing at $t = 0$ and approaching ∞ as $t \to \infty$, it has a minimum ρ at a positive argument $\tau > 0$ such that $\rho = M(\tau) = \inf_t M(t)$. It holds then that $\rho < M(0) = 1$, and since $M(t) > 0$ for all t, $\rho > 0$; thus $0 < \rho < 1$. The rate of convergence result follows from the proof of the Chernoff theorem in [21].

More specifically, (18) implies that

$$P_\delta = \rho^n e^{-\theta_n} \tag{19}$$

where $0 \leq \theta_n \leq \kappa\sqrt{n} - \log a$ for some positive a and κ independent of n [21] (a and κ are related to τ and the derivatives of $M(t)$ evaluated at τ). Thus as $n \to \infty$, $P_\delta \to 0$ with the rate of convergence of $O\left(e^{-\kappa\sqrt{n}}\right)$.

This means any deviation of the throughput of a specific network from its mean scales sub-linearly. Thus the throughput converges to linear growth in probability with the convergence rate of the order $\exp(-\kappa\sqrt{n})$ for some positive constant κ.

Based on this concentration analysis, we can arrive at a stronger result. Not only the average cognitive throughput scales linearly in the number of users, but with high probability, the throughput of the cognitive users in any specific network also scales linearly.

IV. Single Primary User Network with Cognitive Distance-Dependent Power Scaling

In this section, we consider the case in which cognitive transmitters can scale their power according to the distance to a single primary user located at the center of the network. Such a model is relevant when primary transmitters are spread out, or whenever the interference from other primary users is negligible. Suppose that the network consists of a single primary user with the transmitter at the center and the receiver within a radius R_0 ($R_0 > 1$). We choose to center the network at the primary transmitter but not at the receiver based on the assumption that the primary receiver location may not be known to the cognitive users. Again this is appealing in a broadcast scenario, or when the primary receivers are passive device. Then, intuitively, the cognitive transmitters further away from the center may transmit at a higher power without significantly increasing the interference to the primary receiver. We confirm that this cognitive power scaling does not hamper the linear scaling of the average cognitive throughput. Furthermore, it allows D_c, the maximum distance between a cognitive transmitter and its receiver, to grow with the network size.

The transmit power of a cognitive user now scales with the radius r (from the network center) at which this cognitive transmitter is located as in (5), repeated here for convenience:

$$P = P_c r^\gamma$$

where P_c is a constant, and γ is the cognitive power exponent. As shown later, for the interference from the cognitive users to the primary receiver to stay bounded, we require that $\gamma < \alpha - 2$.

A. Interference from cognitive users

Consider the worst-case interference from all cognitive transmitters to a primary or cognitive receiver. Similar to the development in (9) and (10), the worst-case interference is upper-bounded by the interference to the network center. Specifically, the *average* of this worst-case interference is now

$$\begin{aligned}\bar{I}_n(\gamma) &= 2\pi\lambda P_c \int_\epsilon^R r r^\gamma r^{-\alpha} dr \\ &= \frac{2\pi\lambda P_c}{\alpha - 2 - \gamma}\left(\frac{1}{\epsilon^{\alpha-2-\gamma}} - \frac{1}{R^{\alpha-2-\gamma}}\right).\end{aligned} \tag{20}$$

If $\gamma < \alpha - 2$, then $\bar{I}_n(\gamma)$ is bounded as $R \to \infty$. Thus we can let the cognitive users transmit with higher power the further they are from the primary user, provided that their power scaling satisfies $\gamma < \alpha - 2$. This power scaling is dependent on the propagation environment.

With $\gamma < \alpha - 2$, as $n \to \infty$, the worst-case average interference from all cognitive users approaches

$$\bar{I}_\infty(\gamma) = \frac{2\pi\lambda P_c}{\alpha - 2 - \gamma}\frac{1}{\epsilon^{\alpha-2-\gamma}}. \tag{21}$$

This can be interpreted as the interference without power scaling but with an effective path loss decreased to $\alpha - \gamma$. Therefore, the power scaling of the cognitive users effectively takes the advantage of a high path loss environment, when signal power decays with an exponent faster than 2.

B. The effect of cognitive power scaling on D_c

Recall that the maximum distance between a cognitive transmitter and its receiver is D_c. The cognitive channel gain then satisfies $|h_{ii}|^2 \geq 1/D_c^\alpha$. Thus the received power at the cognitive receiver is lower bounded as

$$P_r = P|h_{ii}|^2 \geq P_c\frac{r^\gamma}{D_c^\alpha}. \tag{22}$$

Consider the ratio r^γ/D_c^α. If we constrain this ratio to be at least a constant, then D_c can grow with distance such that

$$D_c \leq K_d r^{\gamma/\alpha} \tag{23}$$

for some constant value K_d. This growth in D_c still guarantees a minimum received power at a cognitive receiver. Therefore, when we let the transmit power of cognitive users scale with distance, the maximum distance between a cognitive Tx and its Rx can also scale with distance. Noting that $\gamma < \alpha - 2$, for $r > 1$, we have

$$D_c \leq K_d r^{\gamma/\alpha} < K_d r^{1-2/\alpha}. \tag{24}$$

The radius r grows at the same rate with n. Thus depending on the power scaling exponent γ, the cognitive Tx-Rx distance can grow with an exponent of up to $1 - 2/\alpha$. For a large α, this growth is almost at the same rate as that of the network.

C. Cognitive throughput scaling

We now examine the throughput scaling of the cognitive users as $n \to \infty$, given the maximum cognitive Tx-Rx distance grows as in (23) with equality. Again we consider lower and upper bounds to the average cognitive throughput.

1) Lower bound on the average cognitive throughput: Consider a cognitive receiver, the interference from the primary user will be bounded because of the finite primary transmit power and the cognitive receiver protected radius ϵ. Again denote this primary-interference upper bound as $I^p_{\max}$. The interference from the other cognitive users is also bounded as in (20) and (21).

The rate of a cognitive user in (3) can now be written as

$$C_i = \log\left(1 + \frac{P_c r^\gamma |h_{ii}|^2}{I_i^p + I_i^c + \sigma^2}\right). \tag{25}$$

Since $I_i^p \leq I^p_{\max}$, together with the bounds on D_c in (23) and on the received power in (22), we now have

$$C_i \geq \log\left(1 + \frac{P_c}{(I^p_{\max} + \sigma^2 + I_i^c)K_d^\alpha}\right). \tag{26}$$

Again applying the Jensen inequality, the average transmission rate of each cognitive user satisfies

$$E[C_i] \quad \geq \quad \log\left(1 + \frac{P_c}{(I^p_{\max} + \sigma^2 + E[I_i])K_d^\alpha}\right).$$

As $n \to \infty$, this lower bound approaches a constant as

$$E[C_i] \geq \log\left(1 + \frac{P_{r,\min}}{\left[I^p_{\max} + \sigma^2 + \bar{I}_\infty(\gamma)\right]K_d^\alpha}\right) \tag{27}$$

where $\bar{I}_\infty(\gamma)$ is given in (21). Thus the average per-user capacity of the cognitive users in this network also remains at least a constant as $n \to \infty$. The average throughput of the cognitive users hence grows at least linearly with n.

2) Upper bound on the average cognitive throughput: The maximum capacity of each cognitive user is achieved when the transmitter and its receiver are within the minimum distance ϵ from each other. Taking into account the power scaling, the capacity averaged over all spatial locations of user i satisfies

$$E[C_i] \leq \int_\epsilon^R \log\left(1 + \frac{P_c r^\gamma \epsilon^{-\alpha}}{\sigma^2 + I_i^p + I_i^c}\right)\frac{2r}{R^2 - \epsilon^2}dr. \tag{28}$$

Note the following integration, in which $a > 0$ is a constant:

$$\begin{aligned}\int_\epsilon^R \log(1 + ar^\gamma)2r dr &= r^2 \log(1 + ar^\gamma)\Big|_\epsilon^R - \int_\epsilon^R r^2 \frac{a\gamma r^{\gamma-1}}{1 + ar^\gamma}dr \\ &\leq r^2 \log(1 + ar^\gamma)\Big|_\epsilon^R\end{aligned}$$

for $\gamma > 0$. Apply this inequality to (28), as $n \to \infty$, since $R = \Theta(\sqrt{n})$,

$$\begin{aligned}E[C_i] &\leq \frac{1}{R^2 - \epsilon^2}\left[r^2 \log(1 + ar^\gamma)\right]\Big|_\epsilon^R \\ &= \Theta(\gamma \log(R)) = \Theta(\log(n)),\end{aligned} \tag{29}$$

which implies that this upper-bound on the average per-user cognitive capacity grows as $\log(n)$. Thus with positive power scaling ($\gamma > 0$), the throughput of the cognitive users grows at most super-linearly as $n \log(n)$.

3) Average cognitive throughput scaling: Combining the lower bound (27) and upper bound (29), we can conclude that with positive power scaling, the *average* throughput of the cognitive users scales at least linearly with n and at most as $n \log(n)$.

V. Conclusion

In this paper, we have determined the throughput scaling of a network of cognitive users simultaneously communicating in the presence of a single or multiple primary users. Using single-hop transmission, we show that, as the number of cognitive users n increases ($n \to \infty$), the average throughput of the cognitive users scales at least *linearly* with n. If the cognitive users transmit with a constant power, provided the cognitive transmitter-receiver distance is bounded, then any network realization can achieve this linear throughput scaling with high probability. Moreover, this linear throughput scaling holds in the presence of multiple primary users. On the other hand, if the cognitive users can scale their power according to the distance from a single, designated primary user, then the average network throughput scales at least linearly with n and at most as $n\log(n)$. In this case, the transmitter-receiver distance of a cognitive user can grow with the network size up to a feasible exponent dependent on the path loss factor. Our results suggest that an initial approach to building a scalable cognitive network can be to employ only single-hop transmission for the cognitive users. This single-hop transmission eliminates routing and appears suitable for cognitive devices often of an opportunistic nature.

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APPENDIX
BOUNDS ON THE INTERFERENCE FROM THE PRIMARY USERS

To show that I_A in (6) is bounded, we use the following inequalities:

$$\frac{1}{(x^2+y^2)^{\alpha/2}} \leq \frac{2^{\alpha/2}}{(x+y)^{\alpha}}\ , \quad x+y>0$$

and

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{(ak+b)^{\alpha}} &\leq \frac{1}{b^{\alpha}} + \int_0^{\infty}\frac{1}{(ax+b)^{\alpha}}dx\ , \quad b>0\\ &= \frac{1}{b^{\alpha}} + \frac{1}{(\alpha-1)a}\frac{1}{b^{\alpha-1}}\ , \quad b>0.\end{aligned}$$

Applying these inequalities to the following generic sum with $a>0$, $c>0$ and $b+d>0$ as:

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{\left[(ak+b)^2+(cm+d)^2\right]^{\alpha/2}} \tag{30}$$

$$\leq \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{2^{\alpha/2}}{(ak+b+cm+d)^{\alpha}}$$

$$\leq 2^{\alpha/2}\sum_{m=0}^{\infty}\left(\frac{1}{(cm+b+d)^{\alpha}} + \frac{1}{(\alpha-1)a}\frac{1}{(cm+b+d)^{\alpha-1}}\right)$$

$$\begin{aligned}\leq 2^{\alpha/2}\Bigg[&\frac{1}{(b+d)^{\alpha}} + \frac{1}{(\alpha-1)c}\frac{1}{(b+d)^{\alpha-1}} + \\ &\frac{1}{(\alpha-1)a}\left(\frac{1}{(b+d)^{\alpha-1}} + \frac{1}{(\alpha-2)c}\frac{1}{(b+d)^{\alpha-2}}\right)\Bigg].\end{aligned} \tag{31}$$

Thus for $\alpha>2$, this summation is bounded.

Now consider I_A in (6). Denote I_{A1} as the first double-summation, then it can be rewritten as

$$I_{A1} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{\left[\left(2\sqrt{3}k-\epsilon\cos\theta\right)^2+\left(2m-\epsilon\sin\theta\right)^2\right]^{\alpha/2}}$$

$$+\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{\left[\left(2\sqrt{3}k+\epsilon\cos\theta\right)^2+\left(2m+\epsilon\sin\theta\right)^2\right]^{\alpha/2}} - \frac{1}{\epsilon^{\alpha}}$$

Since $\epsilon<1$ (normalized to R_0), then $|\epsilon\cos\theta|\leq 1$ and $|\epsilon\sin\theta|\leq 1$ for all $\theta\in[0,2\pi]$. For each of the double-summations in I_{A1}, after separating out the first three finite terms corresponding to $(k=0, m=0)$, $(k=0, m=1)$, and $(k=1, m=0)$, then the rest can be rewritten in form (30) with $b+d>0$, which is bounded. Similarly for the second summation in I_A (6). Therefore for any θ, I_A in (6) is bounded.

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